Received: *November 22, 2000*, Accepted: *January 26, 2000*
Revised: *January 25, 2001*

Orbifold boundary states from Cardy's condition

Marco Billó
Dipartimento di Fisica Teorica, Università di Torino and
I.N.F.N., Sezione di Torino, via P. Giuria 1, I-10125, Torino, Italy
E-mail: billo@to.infn.it

Ben Craps
Enrico Fermi Institute, University of Chicago
*5640 S. Ellis Av., Chicago, IL 60637, USA**
Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven
B-3001 Leuven, Belgium
E-mail: craps@theory.uchicago.edu

Frederik Roose
Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven
B-3001 Leuven, Belgium
Institut Henri Poincaré, 11 rue Pierre et Marie Curie
F-75231 Paris Cedex 05, France
E-mail: Frederik.Roose@fys.kuleuven.ac.be

JHEP01(2001)038

Abstract: Boundary states for D-branes at orbifold fixed points are constructed in close analogy with Cardy's derivation of consistent boundary states in RCFT. Comments are made on the interpretation of the various coefficients in the explicit expressions, and the relation between fractional branes and wrapped branes is investigated for $\mathbb{C}^2/\Gamma$ orbifolds. The boundary states are generalised to theories with discrete torsion and a new check is performed on the relation between discrete torsion phases and projective representations.

Keywords: D-branes, Conformal Field Models in String Theory.

*Present address

Contents

1. Introduction

String theory is often well-behaved on classically singular spaces. As an example, consider type-IIA string theory compactified on a K3 manifold. When a two-sphere of the K3 collapses to zero volume, the resulting space is singular. Duality with the heterotic string relates this singularity to the appearence of an extended gauge symmetry [1, 2]. The additional massless gauge bosons are naturally interpreted as

D2-branes wrapping the vanishing two-sphere. This phenomenon is non-perturbative in string theory, i.e., it is beyond reach of conformal field theory (CFT).

Close to the singularity, the degenerate K3 manifold looks like $\mathbb{C}^2/\Gamma$, with Γ a discrete subgroup of SU(2). String theory on $\mathbb{C}^2/\Gamma$ can be described by an orbifold CFT [3, 4]. The difference with the previous paragraph, where there was no CFT description, is that the orbifold automatically breaks the extended gauge symmetry by a B-flux through the vanishing cycles [5, 6]. The wrapped D2-branes appear as fractional D0-branes in the orbifold theory [7, 8].

The boundary state formalism [9]–[12] (see, for instance, [13]–[16] for recent reviews) is a powerful framework for studying D-branes. D-branes were originally introduced in string theory as hyperplanes on which open strings can end [17, 18, 19]. Boundary states provide a closed string description of D-branes. Consistency with the open string description [20] leads to severe constraints on the possible boundary states in a given string theory. These constraints were analysed by Cardy [21] for a class of rational conformal field theories (RCFT), and have become known as "Cardy's condition". A careful study of this condition allowed Cardy to explicitly construct a set of consistent boundary states.

In this paper we derive boundary states for fractional D-branes at an orbifold fixed point in close analogy with Cardy's construction. The result generalises the boundary states of [22]–[26]. The main advantage of our approach is that it makes the origin of the various coefficients in the expressions for the boundary states manifest. For instance, for $\mathbb{C}^2/\Gamma$ orbifolds, typical sine factors arise from the modular transformation properties of chiral blocks. We show how these transformation properties are consistent with closed string one-loop modular invariance for both compact and non-compact orbifolds.

Some subtleties appear in the Cardy-like construction of the fermionic parts of boundary states, even in flat space. Using observations in [27, 28, 29] we show that the analogy with Cardy's construction is pretty close, though. For instance, the boundary states for "electric" and "magnetic" type-0 D-branes [30, 31, 26] come out very naturally.

We elaborate on the correspondence between fractional and wrapped branes for $\mathbb{C}^2/\Gamma$ orbifolds. For more general orbifolds, the wrapped brane picture would only be valid when the orbifold is blown up to large volume, but in the case we are considering there is enough supersymmetry for the picture to make sense for the undeformed orbifold as well. We determine the explicit basis transformation that takes one from the twisted Ramond-Ramond potentials to the potentials associated to the vanishing cycles.

Recently [32]–[36] there has been a lot of interest in orbifolds with discrete torsion [37]. Discrete torsion means that the action of the orbifold group in twisted sectors is modified by certain phase factors, "discrete torsion phases". In [38] it was argued that D-branes in these theories are associated to projective representations of the orbifold group, where the factor system of the projective representations is related to the discrete torsion phases. We write down boundary states for fractional branes at a fixed point of an orbifold with discrete torsion (extending results of [22]). We use these boundary states to check the consistency of the relation between factor systems and discrete torsion phases for D-branes localised at a fixed point.[1]

The paper is organised as follows. Section 2 contains a brief review of Cardy's condition and the construction of consistent boundary states in RCFT. In section 3 we construct boundary states for D-branes in flat space. Section 4 deals with D-branes at orbifold fixed points. First we briefly review the general structure of orbifold theories. Next we study modular transformation properties of twisted Virasoro characters and write down modular invariant partition functions for both compact and non-compact orbifolds. Then we construct boundary states for fractional branes at orbifold fixed points. For $\mathbb{C}^2/\Gamma$ orbifolds we comment on the relation to McKay correspondence [39] and on the relation with wrapped branes. In section 5 we study D-branes at fixed points of discrete torsion orbifolds. Appendix A discusses the simple example of the free bosons in flat space. Appendix B contains details on chiral blocks and their modular transformation properties. In appendix C we have collected some information about theta functions and discrete groups.

2. Cardy's condition

In this section, Cardy's construction of consistent boundary states is briefly reviewed. The original prescription was derived in the context of a rational conformal field theory on a cylinder. It may be considered as a direct implementation of open-closed consistency. We closely follow the original derivation in [21]. As it turns out, the procedure can be generalised in specific instances of non-rational CFTs. We defer the details of that to the following sections, however.

2.1 Rational CFT

Rational CFT is a realisation of holomorphic and anti-holomorphic symmetry algebras $\mathcal{A}_L$ and $\mathcal{A}_R$, both containing the Virasoro algebra. The number of primaries in a rational CFT is finite. We label these primaries by an index i. The Virasoro characters in the corresponding irreducible modules are

$$\chi_i(q) = \mathrm{Tr}_i\, q^{L_0 - c/24}\,, \tag{2.1}$$

where $q = e^{2\pi \mathrm{i}\tau}$, with τ the modular parameter. We shall only consider purely imaginary τ in this paper.

[1]Note, however, that our analysis is only valid for D-branes localised at a fixed point. For more general D-branes there is no such straightforward relation between discrete torsion phases and projective representations [33].

One identifies a sub-algebra $\mathcal{A}$ of both $\mathcal{A}_L$ and $\mathcal{A}_R$, and denotes the corresponding generators by $W^{(r)}$ and $\tilde{W}^{(r)}$. The choice of $\mathcal{A}$ will largely determine what kind of D-branes (open strings, boundary states) one wishes to keep in the theory. At world-sheet boundaries the generators of the holomorphic and the anti-holomorphic embeddings of $\mathcal{A}$ are related through gluing conditions. In the rest of this section, it is assumed that the "preserved" symmetry algebra $\mathcal{A}$ coincides with the algebras $\mathcal{A}_L$ and $\mathcal{A}_R$, which are thus isomorphic. In this section we only consider theories with diagonal partition function on the torus, i.e.,

$$Z(q,\bar{q}) = \sum_j \chi_j(q)\,(\chi_j(q))^* \,. \tag{2.2}$$

Consider the cylinder amplitude with boundary conditions labeled by α and β. In the loop channel, one has

$$Z_{\alpha\beta} = \sum_i n^i_{\alpha\beta}\chi_i(q)\,, \tag{2.3}$$

where the integers $n^i_{\alpha\beta}$ denote the multiplicities of the representations i running in the loop. In the tree channel, this amplitude reads

$$Z_{\alpha\beta} = \langle\alpha|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-c/12)}|\beta\rangle\,, \tag{2.4}$$

where $\tilde{q} = e^{-2\pi \mathrm{i}/\tau}$. In the latter equation, the boundary states $\langle\alpha|$ and $|\beta\rangle$ impose the boundary condition

$$\left(W^{(r)}_n - (-)^s\Omega(\tilde{W}^{(r)}_{-n})\right) = 0 \tag{2.5}$$

on the Fourier modes of the symmetry generators; that is,

$$\left(W^{(r)}_n - (-)^s\Omega\left(\tilde{W}^{(r)}_{-n}\right)\right)|\beta\rangle = 0\,, \tag{2.6}$$

where s is the spin of $W^{(r)}$ and Ω an automorphism[2] of the preserved algebra $\mathcal{A}$. In the remaining of this section we assume for simplicity that Ω is the trivial automorphism.

The way to proceed from here is to choose a convenient basis of solutions to eq. (2.6), the so-called Ishibashi states [12]. The consistent boundary states are then built as particular linear combinations of the Ishibashi states. Consider a highest weight representation j of $\mathcal{A}_L$ and the corresponding (isomorphic) representation $\tilde{j}$ of $\mathcal{A}_R$. The states of j are linear combinations of states of the form $\prod_I W^{(r_i)}_{-n_I}|j;0\rangle$, where the $W^{(r_i)}_{-n_I}$ are lowering operators and $|j;0\rangle$ is the highest weight state. Denote the elements of an orthonormal basis of the representation j by $|j;N\rangle$ and the

[2] For instance, in the case of a free boson, choosing Ω the trivial automorphism corresponds to Neumann boundary conditions, while a non-trivial one gives Dirichlet conditions (see appendix A).

corresponding basis of $\tilde{j}$ by $\widetilde{|j;N\rangle}$. In terms of the anti-unitary operator U defined by

$$U\widetilde{|j;0\rangle} = \widetilde{|j;0\rangle}^* ; \qquad U\tilde{W}^{(r_I)}_{-n_I}U^{-1} = (-)^{s_{r_I}}\tilde{W}^{(r_I)}_{-n_I}, \tag{2.7}$$

the states

$$|j\rangle\rangle \equiv \sum_N |j;N\rangle \otimes U\widetilde{|j;N\rangle} \tag{2.8}$$

solve eq. (2.6) (for $\Omega = 1$). These are the Ishibashi states.

2.2 Cardy's condition

The consistent boundary states are linear combinations of the Ishibashi states eq. (2.8):

$$|\alpha\rangle = \sum_j B^j_\alpha |j\rangle\rangle . \tag{2.9}$$

We can now rewrite eq. (2.4) as

$$Z_{\alpha\beta} = \sum_j (B^j_\alpha)^* B^j_\beta \chi_j(\tilde{q}) , \tag{2.10}$$

where

$$\chi_j(\tilde{q}) = \langle\langle j|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-c/12)}|j\rangle\rangle = \mathrm{Tr}_j\, \tilde{q}^{L_0-c/24} , \tag{2.11}$$

which is consistent with eq. (2.1). In deriving eq. (2.10) we have used the fact that the Ishibashi states are orthogonal in the sense that

$$\langle\langle j'|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-c/12)}|j\rangle\rangle = 0 \qquad \text{if } j \neq j' . \tag{2.12}$$

eq. (2.3) is transformed to the tree channel by the modular S transformation:

$$Z_{\alpha\beta} = \sum_{i,j} n^i_{\alpha\beta} S^j_i \chi_j(\tilde{q}) . \tag{2.13}$$

Demanding equality of eq. (2.10) and eq. (2.13) eventually yields

$$\sum_i S^j_i n^i_{\alpha\beta} = (B^j_\alpha)^* B^j_\beta , \tag{2.14}$$

at least if no two representations of the holomorphic algebra have the same Virasoro character. Eq. (2.14) is called Cardy's equation. The requirement that the multiplicities $n^i_{\alpha\beta}$ be non-negative integer numbers is a strong condition on the coefficients B^j_α. This constraint is non-linear: multiplying a consistent boundary state by a non-integer number will generically not yield a consistent boundary state.

2.3 Cardy's solution

In [21], Cardy gave a solution to his equation eq. (2.14). The first consistent boundary that was constructed, is $|\mathbf{0}\rangle$. It is determined by the requirement that $n^i_{00} = \delta^i_0$, that is, the only representation running in the loop channel is the identity representation. From eq. (2.14), such a state satisfies

$$|B^j_0|^2 = S^j_0 \,. \tag{2.15}$$

The entries S^j_0 of the modular transformation matrix are positive [21], so eq. (2.15) is consistent. eq. (2.15) implies

$$|\mathbf{0}\rangle = \sum_j \sqrt{S^j_0}|j\rangle\rangle \tag{2.16}$$

(up to the relative phases of the coefficients, which are not fixed by these considerations). Similarly, one defines boundary states $|\mathbf{l}\rangle$, with the property that $n^i_{0l} = \delta^i_l$, for every primary l. Using eq. (2.14), eq. (2.15) and the fact that $S^j_0 > 0$, these states are seen to be

$$|\mathbf{l}\rangle = \sum_j \frac{S^j_l}{\sqrt{S^j_0}}|j\rangle\rangle \,. \tag{2.17}$$

In Cardy's solution, the multiplicities $n^i_{\alpha\beta}$ coincide with the fusion rule coefficients of the algebra $\mathcal{A}$. The boundary states eq. (2.17) solve eq. (2.14) because of Verlinde's formula [40, 21].

3. D-branes in flat space

Cardy's construction of consistent boundary states can be made explicit is the important example of the CFT of free bosons. Since this is well-known, we defer most of the discussion to appendix A. Implementing Cardy's construction for free fermionic theories is somewhat less trivial. We shall show that, despite some subtleties, it is possible to follow Cardy's prescription rather closely. This proves to be an elegant way to construct directly the type-II or type-0 boundary states satisfying the requirements of open-closed consistency. We also briefly discuss the way in which the proper normalisations of the boundary states in string theory (as opposed to CFT) are determined.

3.1 Boson boundary state

The basics of the discussion of boundary states in free bosonic theories in terms of Cardy's condition were given already in [41, 42]. In this subsection we give a very brief summary. For completeness, and also in order to establish notations and conventions, details are collected in appendix A.

JHEP01(2001)038

On each spacetime coordinate we can impose either Neumann or Dirichlet boundary conditions, corresponding to a trivial ($\Omega = 1$) or non-trivial ($\Omega = -1$) choice of the automorphism Ω. Correspondingly, the boundary states can be of Neumann or Dirichlet type in each spacetime direction. As an example, the consistent boundary state describing a D-brane localised at a position x along a circle with radius R is given by

$$|x\rangle_D = \left(\frac{\sqrt{\alpha'}}{\sqrt{2}R}\right)^{1/2} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}x\frac{k}{R}} |(k,0)\rangle\rangle_D \,, \tag{3.1}$$

where the Ishibashi states $|(k,0)\rangle\rangle_D$ take the form

$$|(k,0)\rangle\rangle_D = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\alpha_{-n}\tilde{\alpha}_{-n}\right) |(k,0)\rangle \,. \tag{3.2}$$

String theory. The introduction of two boundaries at fixed positions necessarily introduces a dimensionful scale in the theory. In the resulting CFT on a strip, this scale is the width of the strip. The context where Cardy first derived his condition and solutions was in CFT applications to statistical mechanical systems, where the scale has a physical significance. As such, equality is demanded of expressions eq. (2.10) and eq. (2.13) on a *fixed* cylinder. This situation is to be contrasted with string theory. Let us adopt the path-integral point of view for a while. Given a topology, the prescription to obtain the string amplitude is then that one should integrate over all metrics and divide out the volume of the gauge group, i.e., the (Diff x Weyl) group. In the present case, this implies the presence of an integral over the modulus of the cylinder in one-loop open string amplitudes. In the closed string channel, this is reflected by the integral in the closed string propagator below. We shall now indicate how equality of the amplitudes in the open and closed string channels fixes the normalisation of the boundary states.

Let us consider strings propagating in flat space. For boundary conditions $1,2$ corresponding to parallel p-branes located at y_1, y_2, we have

$$\mathcal{Z}_{1,2} = V_{p+1} \int_0^{\mathrm{i}\infty} \frac{d\tau}{\tau}\, Z^{(d)}_{1,2}(q) \,, \tag{3.3}$$

where $q = \exp(2\pi\mathrm{i}\tau)$, $d = 26$ for the bosonic string or $d = 10$ for the superstring and $Z^{(d)}_{1,2}(q)$ is the standard open string partition function in d flat directions, of which $p+1$ are Neumann (see for instance [45]), with the prescribed boundary conditions. Notice that we included in $Z^{(d)}_{1,2}(q)$ the integration over the momentum along the Neumann directions. The partition function $Z^{(d)}_{1,2}(q)$ can be expressed in terms of the Neumann and Dirichlet open string characters introduced in the previous section, in the decompactified limit $R \to \infty$; notice however that ghost contributions effectively cancel those of two Dirichlet directions. In the superstring case, the contributions of the fermionic sectors, to be considered shortly in section 3.2, and of super-ghosts, must obviously be added. Upon modular transformation, the open string amplitude $\mathcal{Z}_{1,2}$ has to be reinterpreted in terms of closed string propagation between boundary states:

$$\langle B_1| \frac{\alpha'}{4\pi} \int_{|z|<1} \frac{d^2 z}{|z|^2} z^{L_0 - a} \bar{z}^{\tilde{L}_0 - \tilde{a}} |B_2\rangle \,. \tag{3.4}$$

Here we used the canonical normalization of the closed string propagator. The boundary states $|B\rangle$ that do the job are given by the product of consistent boundary states of Neumann type, eq. (A.7), for $p+1$ directions, and of states of Dirichlet type, eq. (A.11), for $d-(p+1)-2$ directions (the -2 accounts for the ghost contributions), times an overall normalisation. In the decompactified limit for all directions, the appropriate overall normalisation[3] is given (see for instance [13]) by

$$N_p = \frac{\sqrt{\pi}}{2} 2^{\frac{10-d}{4}} (2\pi\sqrt{\alpha'})^{\frac{d-4}{2}-p} \,. \tag{3.5}$$

The corresponding tension of the Dp-brane, as computed from the graviton-dilaton exchange in the field-theory limit of the closed string amplitude [19] is T_p/κ, with $T_p = 2N_p$ and κ being the gravitational coupling constant.

3.2 Fermion boundary state

Let us next focus on the world-sheet fermions. In principle, these may be treated in the covariant formulation [46], similarly to the bosons above. For simplicity, however, we opt for a light-cone treatment of the fermionic sector. Apart from one subtle point explained shortly, there is no obvious gain for our purposes in the covariant formulation. The fermions thus realise the SO(8) affine algebra $\mathcal{A}$ at level 1. In both the NS- and R-sectors, the unprojected Fock space decomposes into two irreducible modules $\mathcal{H}^{\pm}$ of $\mathcal{A}$, according to the GSO projection. The corresponding four inequivalent representations of the SO(8) algebra $\mathcal{A}$ are the singlet (o) and the vector (v) in the NS sector, the spinor (s) and conjugate spinor (c) in the R sector. Their characters are $\chi_a(q) = \mathrm{Tr}_a(q^{L_0 - 1/6})$, where $a = o, v, s, c$.

We put an extra minus sign in the definition of the spinorial characters s, c compared to their natural definition as SO(8) characters. This peculiar choice actually implements the space-time spin-statistics [28, 27, 29]. With this choice, the $\mathrm{SO}(2n)_1$ characters are given explicitly by

$$\begin{aligned} \chi_v &= \frac{1}{2}\left[\left(\frac{\theta_3}{\eta}\right)^n - \left(\frac{\theta_4}{\eta}\right)^n\right], & \chi_o &= \frac{1}{2}\left[\left(\frac{\theta_3}{\eta}\right)^n + \left(\frac{\theta_4}{\eta}\right)^n\right], \\ \chi_s &= -\frac{1}{2}\left[\left(\frac{\theta_2}{\eta}\right)^n + \left(\frac{\theta_1}{\eta}\right)^n\right], & \chi_c &= -\frac{1}{2}\left[\left(\frac{\theta_2}{\eta}\right)^n - \left(\frac{\theta_1}{\eta}\right)^n\right], \end{aligned} \tag{3.6}$$

where, in fact, $\theta_1(\tau)$ is zero. For now, our primary interest is in the $n=4$ case.

[3] For commodity, we collect in this normalisation also the pre-factors of powers of α' and 2 that in the previous section appeared in the consistent states eq. (A.7), eq. (A.11).

JHEP01(2001)038

The modular T and S transformations are encoded in the following matrices acting on the character vector χ_a (ordered as v, o, s, c)

$$\begin{aligned} T_{(2n)} &= \mathrm{diag}\left(-\mathrm{e}^{-n\pi\mathrm{i}/12}, \mathrm{e}^{-n\pi\mathrm{i}/12}, \mathrm{e}^{n\pi\mathrm{i}/6}, \mathrm{e}^{n\pi\mathrm{i}/6}\right), \\ S_{(2n)} &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & (-\mathrm{i})^n & -(-\mathrm{i})^n \\ 1 & -1 & -(-\mathrm{i})^n & (-\mathrm{i})^n \end{pmatrix}. \end{aligned} \tag{3.7}$$

The form of $S_{(2n)}$ suggests that it is the v class rather than the singlet class o, that plays the role of the identity in the fusion rules. This exchange has been explained in the covariant formulation, where one considers the combined system of $\mathrm{SO}(1,9)$ fermions and super-ghosts [27]. Indeed, thinking of fusion rules, it is logical that a representation with even world-sheet spinor number should play the role of the identity representation.

The bulk conformal field theory is specified by a choice of modular invariant. To make the analogy with section 2 as close as possible, we shall first consider a diagonal partition function. The torus partition function for this theory is

$$Z^{0\mathrm{B}}(q) \propto \left(|\chi_o(q)|^2 + |\chi_v(q)|^2 + |\chi_s(q)|^2 + |\chi_c(q)|^2\right), \tag{3.8}$$

i.e., the following (left,right) sectors are kept: $(NS+, NS+) \oplus (NS-, NS-) \oplus (R+, R+) \oplus (R-, R-)$. The contribution of the bosonic string fields completes this expression to the full type-0B torus partition function.

Starting from this bulk CFT, boundaries may be introduced in string world-sheets. Below attention will be focused on D9-branes,[4] preserving the full Lorentz, whence $\mathrm{SO}(8)$ invariance in light-cone gauge. The Dp-branes with $p < 9$ are obtained by T-duality.

Let us first construct the Ishibashi states. In both the NS and R sectors, two gluing conditions are possible:

$$\psi_r = \mathrm{i}\eta\tilde{\psi}_{-r}, \tag{3.9}$$

where $\eta = \pm 1$. States solving these conditions as in eq. (2.8) are

$$|\sigma; \eta\rangle\rangle = \prod_\mu \exp\left[\mathrm{i}\eta \sum_{r>0} \psi^\mu_{-r}\tilde{\psi}^\mu_{-r}\right] |\sigma, \eta; 0\rangle\rangle, \tag{3.10}$$

with $\sigma = \mathrm{NS}, \mathrm{R}$ indicating the NS-NS or R-R sector and μ running on the transverse directions. In the R-R (NS-NS) sector, the mode numbers r are (half-)integer. Also,

[4] Of course, strictly speaking it is inconsistent to consider a D9-brane without the corresponding anti-brane, because this would lead to a R-R tadpole. Moreover, for D9 branes (or D8's in type-IIA) one actually needs a covariant formulation, as the exact cancellation of the ghost-superghost contributions against two coordinates takes place for Dirichlet directions. This subtlety shows up, for instance, in systems with eight ND directions, like the D0-D8 one [46].

there is a non-trivial zero-mode part in the R sector:

$$|\mathrm{R},\eta;0\rangle\rangle = \mathcal{M}^{(\eta)}_{AB}\,|A\rangle|\tilde{B}\rangle\,, \tag{3.11}$$

where

$$\mathcal{M}^{(\eta)} = C\Gamma^0\Gamma^{l_1}\cdots\Gamma^{l_p}\left(\frac{1+\mathrm{i}\eta\Gamma_{11}}{1+\mathrm{i}\eta}\right), \tag{3.12}$$

with C being the charge conjugation matrix and l_i labeling the space directions of the D-brane world volume. The vacuum states $|A\rangle|\tilde{B}\rangle$ for the fermionic zero-modes ψ_0^μ and $\tilde{\psi}_0^\mu$ transform in the 32-dimensional Majorana representation.

From the states in eq. (3.10) a proper set of Ishibashi states is provided by taking appropriate linear combinations:

$$|v\rangle\rangle = \frac{1}{2}\left(|\mathrm{NS},+\rangle\rangle - |\mathrm{NS},-\rangle\rangle\right); \tag{3.13}$$

$$|o\rangle\rangle = \frac{1}{2}\left(|\mathrm{NS},+\rangle\rangle + |\mathrm{NS},-\rangle\rangle\right); \tag{3.14}$$

$$|s\rangle\rangle = \frac{1}{2}\left(|\mathrm{R},+\rangle\rangle + |\mathrm{R},-\rangle\rangle\right); \tag{3.15}$$

$$|c\rangle\rangle = \frac{1}{2}\left(|\mathrm{R},+\rangle\rangle - |\mathrm{R},-\rangle\rangle\right). \tag{3.16}$$

They are mutually orthogonal states satisfying the type-0B GSO projection. The above labelling follows the general convention of eq. (2.11); the corresponding chiral blocks may indeed be verified to be:

$$\langle\langle m|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-c/12)}|n\rangle\rangle = \delta_{mn}\,\chi_m(\tilde{q})\,, \tag{3.17}$$

with $m,n = o,v,s,c$.

From the Ishibashi states we wish to construct a set of consistent boundary states $|a\rangle$, with $a = v,o,s,c$, that is, we are after a consistent set of coefficients B_a^j of eq. (2.9). These are obtained by Cardy's formula eq. (2.17), with the S matrix of eq. (3.7). As alluded to before, the fact that $S_v^m = 1/2 > 0$ for all m reflects the fact that it is the v representation that is the analogue of the identity representation in RCFT. As such, $|v\rangle$ is the analogue of the state $|\mathbf{0}\rangle$ in section 2.3. We obtain

$$\begin{aligned}
|v\rangle &= \frac{1}{\sqrt{2}}\sum_m |m\rangle\rangle = \frac{1}{\sqrt{2}}(|v\rangle\rangle + |o\rangle\rangle + |s\rangle\rangle + |c\rangle\rangle) \\
&= \frac{1}{\sqrt{2}}(|NS,+\rangle\rangle + |R,+\rangle\rangle)\,, \\
|o\rangle &= \sqrt{2}\sum_m (S_{(8)})_v^m\,|m\rangle\rangle = \frac{1}{\sqrt{2}}(|v\rangle\rangle + |o\rangle\rangle - |s\rangle\rangle - |c\rangle\rangle) \\
&= \frac{1}{\sqrt{2}}(|NS,+\rangle\rangle - |R,+\rangle\rangle)\,,
\end{aligned}$$

$$
\begin{aligned}
|s\rangle &= \sqrt{2}\sum_m (S_{(8)})_s^m |m\rangle\rangle = \frac{1}{\sqrt{2}}(|v\rangle\rangle - |o\rangle\rangle + |s\rangle\rangle - |c\rangle\rangle) \\
&= \frac{1}{\sqrt{2}}(-|NS,-\rangle\rangle + |R,-\rangle\rangle)\,, \\
|c\rangle &= \sqrt{2}\sum_m (S_{(8)})_s^m |m\rangle\rangle = \frac{1}{\sqrt{2}}(|v\rangle\rangle - |o\rangle\rangle - |s\rangle\rangle + |c\rangle\rangle) \\
&= \frac{1}{\sqrt{2}}(-|NS,-\rangle\rangle - |R,-\rangle\rangle)\,.
\end{aligned} \tag{3.18}
$$

These states are the type-0B boundary states that may be found in the literature [30, 31, 26]. The states $|v\rangle$ and $|o\rangle$ are usually referred to as electric D9-brane and anti-D9-brane, respectively, while $|s\rangle$ and $|c\rangle$ are called magnetic D9-brane and anti-D9-brane.

From the Cardy construction the integer multiplicities in the loop channel n_{ab}^c follow from the Verlinde formula [40]:

$$
n_{ab}^c = \sum_m \frac{(S_{(8)})_a^m\,(S_{(8)})_b^m\,((S_{(8)})_c^m)^*}{(S_{(8)})_v^m} = 2\sum_m (S_{(8)})_a^m\,(S_{(8)})_b^m\,(S_{(8)})_c^m\,, \tag{3.19}
$$

where the second equality makes use of the explicit form of the matrix $S_{(8)}$ of eq. (3.7). They correspond to the fusion rules

$$
v\otimes\alpha=\alpha\,,\qquad \alpha\otimes\alpha=v\,,\qquad o\otimes s=c\,,\qquad o\otimes c=s\,,\qquad s\otimes c=o\,, \tag{3.20}
$$

that is we have $n_{v\alpha}^i=\delta_\alpha^i$, $n_{\alpha\alpha}^i=\delta_v^i$, $n_{oc}^j=\delta_s^i$, and so on. These fusion rules correspond to the algebra of the conjugacy classes o, v, s, c of SO(8) representations where o and v have been exchanged [28, 27, 29].

Next we want to consider the supersymmetric type-IIB theory, which has the one-loop partition function

$$
Z^{\mathrm{IIB}}(q) \propto |\chi_v(q)+\chi_s(q)|^2\,. \tag{3.21}
$$

As is well known (see, e.g., [46]), of the Ishibashi states (3.14)–(3.16) only $|v\rangle\rangle$ and $|s\rangle\rangle$ are kept after the type-IIB GSO projection $(1+(-)^{F_L})/2\ \times(1+(-)^{F_R})/2$. It is no longer possible to apply Cardy's procedure straightaway; however, the boundary states ensuring open-closed consistency may be obtained in a similar spirit. First reorganise the characters running in the loop channel into the vector $\hat\chi_A$ $(A=0,1,2,3)$ where

$$
\hat\chi_\alpha \equiv (\chi_v+\chi_s, \chi_o+\chi_c, \chi_v-\chi_s, \chi_o-\chi_c) \tag{3.22}
$$

and reorder those in the tree channel into $\hat\chi_M=(\chi_v,\chi_s,\chi_o,\chi_c)$. With respect to these new bases, the S modular transformation is $\hat\chi_A(q)=\hat S_A^M\,\hat\chi_M(\tilde q)$, with

$$
\hat S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{3.23}
$$

JHEP01(2001)038

The matrix $\hat{S}$ is thus seen to commute with the GSO projection, which is just the projector on the upper left block. As such, it makes sense to just restrict ourselves to the GSO projected Ishibashi states $|\hat{M}\rangle\rangle$, with $\hat{M} = v, s$, and introduce consistent states $|\hat{A}\rangle, (\hat{\alpha} = 0, 1)$ given by a projected version of Cardy's formula eq. (2.17) (notice that $|0\rangle$ corresponds to the identity, with $\hat{S}_0^{\hat{M}} = 1 > 0$):

$$\begin{aligned} |0\rangle &= \sum_{\hat{M}} |\hat{M}\rangle\rangle = |v\rangle\rangle + |s\rangle\rangle \\ &= \frac{1}{2}(|NS,+\rangle\rangle - |NS,-\rangle\rangle + |R,+\rangle\rangle + |R,-\rangle\rangle)\,, \\ |1\rangle &= \sum_{\hat{M}} \hat{S}_1^{\hat{M}} |\hat{M}\rangle\rangle = |v\rangle\rangle - |s\rangle\rangle \\ &= \frac{1}{2}(|NS,+\rangle\rangle - |NS,-\rangle\rangle - |R,+\rangle\rangle - |R,-\rangle\rangle)\,. \end{aligned} \tag{3.24}$$

It may be observed that $|0\rangle$ and $|1\rangle$ coincide with the familiar expressions for IIB D9 and anti-D9 brane boundary states. Amplitudes $Z_{\hat{A}\hat{B}}$ with such boundary conditions are given by

$$Z_{00} = Z_{11} = \hat{\chi}_0 = \chi_v + \chi_s\,, \qquad Z_{01} = Z_{10} = \hat{\chi}_1 = \chi_o + \chi_c\,, \tag{3.25}$$

yielding the $D9$-$D9$ and $D9$-$\bar{D}9$ amplitudes indeed.

This Cardy-like derivation of consistent type-II theory boundary states may appear to be somewhat heuristic.[5] The states that we find in this way automatically enjoy a number of nice (physical) properties of D-branes, though. In particular, pure RR boundary states are absent, and all D-branes couple to gravity (see also [48]). The nice aspect of the construction is in displaying these features very explicitly.

Above, only collections of (parallel) D-branes of the same dimension were considered. Allowing more general configurations involving D-branes of different dimensions, mutual consistency will generically impose additional constraints. As an example, one learns that in type-IIA theory with BPS Dp branes for even p, only non-BPS Dq-branes with odd q can be added consistently (see, for instance, [16]).

Let us finally recall that the complete superstring boundary states are obtained by multiplying the bosonic (Neumann or Dirichlet as appropriated to the various directions) and fermionic boundary states we have been discussing, with the overall normalization N_p described in eq. (3.5), for $d = 10$. The type-II boundary states normalised in this way describe Dp-branes with the tension $T_p/\kappa = 2N_p/\kappa$ as in eq. (3.5)

[5] For one thing, in these cases the characters are not linearly independent: the one-loop vacuum amplitude for BPS D-branes vanishes due to the familiar cancellation between the NS and the R sector contributions. Nevertheless, the derivation of Cardy-like conditions may be justified by requiring that not only the vacuum diagram but also diagrams with vertex operators inserted agree between the open and closed string channels. This is analogous to the use of unspecialised characters in [47].

and a charge density [19]

$$\mu_p = 2\sqrt{2}N_p = \sqrt{2\pi}\left(2\pi\sqrt{\alpha'}\right)^{3-p} \tag{3.26}$$

with respect to the canonically normalised RR $(p+1)$-form A_{p+1}.

4. D-branes at an orbifold fixed point

After the flat space D-branes, let us next examine branes in orbifold spaces of $\mathbb{C}^n/G$ type. Some attention will be paid in particular to the $\mathbb{C}^2/\Gamma$ cases, with Γ a discrete subgroup of SU(2). The latter orbifolds correspond to blown-down ALE spaces, and are local models of degenerate K3 surface geometries. The CFT description of such orbifolds is under control, and one primary aim will be to show how a suitably modified Cardy prescription leads to the correct expressions for consistent D-brane boundary states. We will only discuss fractional D-branes localised at the fixed point of the orbifold. [49, 7, 50, 8]

In subsection 4.1 we briefly review a number of general features of orbifold conformal field theories. After a more detailed discussion of the chiral blocks for orbifolds of free theories and their modular transformation properties (subsection 4.2), subsection 4.3 treats the construction of consistent boundary states for D-branes stuck at orbifold fixed points. Some comments regarding the geometrical significance of these states are made at the end of that subsection. Finally, we briefly comment on the relation with the boundary states constructed in [51, 52], where the relation of ALE spaces with non-compact Gepner models was exploited.

4.1 General features of orbifold theories

Orbifold theories [3] are obtained from a parent conformal field theory by modding out a discrete symmetry group G. That is, G is a finite subgroup of the group of endomorphisms of the algebra $\mathcal{A}_L \otimes \mathcal{A}_R$, preserving the left-right decomposition and commuting with the Virasoro algebra [53].

The partition function of an orbifold CFT on a torus has the following structure:

$$Z = \frac{1}{|G|} \sum_{\substack{g,h\in G \\ [g,h]=e}} Z(g,h)\,. \tag{4.1}$$

More generally, sector-dependent phases $\varepsilon(g|h)$, may be introduced in eq. (4.1) [37]. These numbers adding the $Z(g,h)$ with relative weights reflect the possibility to turn on discrete torsion. The discussion of those cases is deferred until section 5, however.

In eq. (4.1), $Z(g,h)$ represents the partition function evaluated with boundary conditions twisted by h and g along the a- and b-cycles of the torus, respectively. As $aba^{-1}b^{-1}$ is in the trivial π_1 homotopy class, these boundary conditions are only consistent if g and h commute (we use the notation $[g,h] = ghg^{-1}h^{-1}$), whence the restriction on the sum in eq. (4.1).

From the hamiltonian point of view, the structure of the orbifold Hilbert space $\tilde{\mathcal{H}}$ is as follows. First, the original untwisted Hilbert space $\mathcal{H}_0$ is projected onto its G-invariant subspace. Next, for each $h \in G$, a twisted Hilbert space $\mathcal{H}_h$ is introduced, in which the fundamental fields obey periodicity conditions twisted by h. As in the untwisted Hilbert space, only states invariant under the action of the orbifold group G are kept. For non-abelian orbifold groups, and/or in the presence of different fixed points, there are additional complications, to which we now turn.

Take a field $\phi(\sigma, \tau)$ in the sector twisted by $h \in G$ (σ and τ are the world-sheet space and time coorditates). It satisfies

$$\phi(\sigma + 2\pi, \tau) = h\phi(\sigma, \tau)\,, \tag{4.2}$$

where $h\phi$ denotes the image of the field ϕ under the action of h. Now G-invariant states are obtained in two steps. First, any state in $\mathcal{H}_h$ is projected inside $\mathcal{H}_h$ onto its N_h-invariant part, where

$$N_h = \{g \in G | [g, h] = e\} \tag{4.3}$$

is the stabilizer group of h. Next one averages by adding the images of such state in the sectors twisted by the remaining elements in the conjugacy class of h. The final result is a G-invariant state in natural correspondence with the conjugacy class.

How is this related to eq. (4.1)? The term $Z(g, h)$ corresponds to

$$Z(g, h) = \mathrm{Tr}_{\mathcal{H}_h}\; g\, q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - c/24} = \sum_{(j, \bar{j})} \chi^g_{h(j)}(q) \bar{\chi}^g_{h(\bar{j})}(\bar{q})\,. \tag{4.4}$$

Each twisted sector Hilbert space $\mathcal{H}_h$ is decomposed into representations of the chiral algebra and we introduced the corresponding twisted chiral blocks

$$\chi^g_{h(j)}(q) = \mathrm{Tr}_{\mathcal{H}_{h(j)}}\; g\, q^{L_0 - c/24}\,. \tag{4.5}$$

The sum over $g \in N_h$ in eq. (4.1) projects onto N_h-invariant states. Combined with their image states in sectors twisted by elements conjugate to h, states invariant under the full orbifold group G are obtained. As such, eq. (4.1) implements the inclusion of twisted sectors and the projection onto invariant states. In the orbifold CFT the chiral symmetry algebra is the G-invariant subalgebra $\mathcal{A}_0 = \mathcal{A}/G$ of $\mathcal{A}$. The twisted sectors correspond to representations of $\mathcal{A}_0$ that were not representations of $\mathcal{A}$.

If the action of (subgroups of) the orbifold group G admits several fixed points $\phi^{(I)}$, the twisted sectors $\mathcal{H}_h$ contain subsectors $\mathcal{H}_h^{(I)}$ whenever h belongs to the subgroup $G^{(I)}$ fixing $\phi^{(I)}$. The partition function of the orbifold is still given by eq. (4.1), with the understanding that the terms $Z(g, h)$ receive contributions from each point fixed by h. Below, a more explicit notation is introduced in the expression of the partition function. The calligraphic symbol $\mathcal{Z}(g,h)$ will denote the h twisted contribution from a *single* fixed point. The partition function then involves an explicit sum over the fixed points:

$$Z = \frac{1}{|G|} \sum_{\substack{g,h\in G \\ [g,h]=e}} \mathcal{Z}(g,h) + \sum_I \frac{1}{|G^{(I)}|} \sum_{\substack{h\neq e, g\in G^{(I)} \\ [g,h]=e}} \mathcal{Z}^{(I)}(g,h)\,. \tag{4.6}$$

The first term comes from the fixed point implicitly associated to the original twisted sectors eq. (4.2), whereas the second term is due to the other fixed points. As will be pointed out in the next section, it is one-loop modular invariance that necessitates the inclusion of the full set of fixed points.

4.2 Chiral blocks and orbifold partition functions

In this section we outline in some detail how the general features of the previous subsection are realised in the specific instance of orbifolds of free bosons and fermions. Technical details are collected in appendix B.

We want to model (super)string propagation on a flat $2n$-dimensional space with a geometrical action by some discrete group G. The flat space may be either compact (T^{2n}) or non-compact ($\mathbb{C}^n$). The G-action is implemented accordingly on the (super)string fields X and ψ corresponding to these directions. In this subsection, we will only discuss the "internal" orbifold CFT. Quantities referring to the remaining part of the $c = 26$ or $c = 15$ CFT are suppressed.

Since any g and h appearing together in eq. (4.1) commute, they can be simultaneously diagonalised. In the sector (g,h) we can therefore choose a basis of complex fields X^l, $l = 1,\ldots,d$ (and similarly for the ψ's) such that

$$g:\ X^l \mapsto \mathrm{e}^{2\pi\mathrm{i}\nu_{g,l}} X^l \tag{4.7}$$

($\bar{X}^l$ transforms with opposite phases), while the twist by h imposes

$$X^l(\tau,\sigma+2\pi) = \mathrm{e}^{2\pi\mathrm{i}\nu_{h,l}} X^l(\tau,\sigma)\,. \tag{4.8}$$

Analogous conditions apply to the fermionic fields. With the diagonal action eq. (4.7), each single complex direction may be discussed separately. For ease of notation, we henceforth denote the eigenvalues of g,h in this direction by $\nu_g \equiv \nu, \nu_h \equiv \nu'$.

Consider the bosonic fields first. The chiral blocks $\chi_h^{(X)g}$ defined as in eq. (4.5), may be expressed explicitly in terms of theta-functions (see appendix B). Out of these, a particular subset corresponds to the untwisted sector. Only these blocks will appear in the open string partition function where strings are stretched between parallel D-branes at the orbifold point, so we concentrate on them in most of this paper. From appendix B one learns that

$$\widehat{\chi}_e^{(X)\,g}(q) = 2\sin\pi\nu\,\frac{\eta(\tau)}{\theta_1(\nu|\tau)}\,, \tag{4.9}$$

where the hat indicates the omission of the zero-modes. The modular transformation properties of these blocks depend crucially on the presence or absence of the zero-mode contribution in eq. (4.9). An example will be given below to illustrate this. Rather than to deal with the general case, we believe it is more useful to show the peculiar role of the zero-modes in a specific instance.

Let us next turn to the fermions. The relevant objects are the v, o, s, c representations of the SO(2) algebra . The corresponding chiral blocks in eq. (B.5) are labelled accordingly. The untwisted characters are

$$\begin{aligned}(\chi_v)_e^g &= \frac{\theta_3(\nu|\tau) - \theta_4(\nu|\tau)}{2\eta(\tau)}\,,\\ (\chi_o)_e^g &= \frac{\theta_3(\nu|\tau) + \theta_4(\nu|\tau)}{2\eta(\tau)}\,,\\ (\chi_s)_e^g &= \frac{\theta_2(\nu|\tau) - \mathrm{i}\,\theta_1(\nu|\tau)}{2\eta(\tau)}\,,\\ (\chi_c)_e^g &= \frac{\theta_2(\nu|\tau) + \mathrm{i}\,\theta_1(\nu|\tau)}{2\eta(\tau)}\,.\end{aligned} \tag{4.10}$$

In the last two cases, these expressions include the fermionic zero-mode contribution (see appendix B). For explicit expressions of $(\chi_a)_h^g$, with $a = v, o, s, c$ in the generic twisted sectors we refer to appendix B likewise.

Modular transformations. The modular properties of orbifold chiral blocks were considered in ref. [53] in the context of RCFT. The theory of free bosons, in particular in the non-compact case, is not an RCFT; however the chiral blocks are simple expressions in terms of theta-functions and as such the modular transformations are easily derived.

In the RCFT case, the generators $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$ of the modular group of the torus are represented on the chiral blocks χ_h^g as follows [53]:

$$\begin{aligned}\chi_h^g &\xrightarrow{S} \sigma(h|g)\,\chi_g^{h^{-1}}\,;\\ \chi_h^g &\xrightarrow{T} \mathrm{e}^{-\pi\mathrm{i}c/12}\,\tau_h\,\chi_h^{hg}\,.\end{aligned} \tag{4.11}$$

The quantities $\sigma(h|g)$ must be symmetric in h and g. In the case of orbifolds of free bosons and fermions, the modular transformations are studied in appendix B. They turn out to take the form eq. (4.11), with appropriate corresponding quantities $\sigma(h,g)$ and τ_h.

In particular, the untwisted boson characters (without zero mode contributions) $\widehat{\chi}_e^{(X)g}$ transform as follows under the S modular transformation:

$$\widehat{\chi}_e^{(X)\,g}(\tilde{q}) = 2\sin\pi\nu\,\chi_g^{(X)\,e}(q)\,, \tag{4.12}$$

corresponding to

$$\widehat{\sigma}(e|g) = 2\sin\pi\nu\,. \tag{4.13}$$

For the untwisted fermions, the action of the S-modular transformation is encoded in matrices $S(h|g)$ acting on the character vector $(\chi_a)^g_h$, $(a = v, o, s, c)$:

$$(\chi_a)^g_e \stackrel{S}{\longrightarrow} (S_{(2)})_a{}^m \, (\chi_m)^e_g \,, \tag{4.14}$$

where $S_{(2)}$ is given in eq. (3.7).

Bosonic zero modes in an example. We next wish to display the role of the bosonic zero-modes in a concrete example. Along the discussion, we will make a detour to comment on the geometrical meaning of the extra factors $2\sin\pi\nu$ present in the S-modular transformations eq. (4.13) of bosonic untwisted chiral blocks. These peculiar factors appear in the partition function of a compact orbifold, where they count fixed points according to the Lefschetz fixed point theorem. The very same factors $2\sin\pi\nu$ will appear in the coefficients of consistent boundary states for fractional D-branes sitting at the orbifold singularity.

Given its mainly illustrative purposes, the discussion can be restricted to the simple case of one complex dimensional abelian orbifolds; that is, to a $\mathbb{Z}_N$ point group action.

Consider the non-compact case first. Denote the $\mathbb{Z}_N$ generator by g. According to eq. (4.1), the orbifold partition function contains Z_{flat}, all the contributions $Z_{\text{n.c.}}(e, g^\alpha)$ needed to project onto $\mathbb{Z}_N$ invariant states in the untwisted sector. The zero modes (the momenta $\mathbf{k}$) multiply the non-zero mode contribution eq. (4.12) as follows:

$$\begin{aligned} Z_{\text{n.c.}}(e, g^\alpha) &= \int d^2\mathbf{k} \langle \mathbf{k}| g^\alpha q^{\frac{\alpha' \mathbf{k}^2}{2}} |\mathbf{k}\rangle |\widehat{\chi}_e^{(X)\, g^\alpha}|^2 = \int d^2\mathbf{k} \delta^2\left((1-g^\alpha)\mathbf{k}\right) q^{\alpha' \mathbf{k}^2/2} |\widehat{\chi}_e^{(X)\, g^\alpha}|^2 \\ &= \det(1-g^\alpha)^{-1} \, |\widehat{\chi}_e^{(X)\, g^\alpha}|^2 = \frac{|\widehat{\chi}_e^{(X)\, g^\alpha}|^2}{4\sin^2 \pi \frac{\alpha}{N}} = \left| \frac{\eta(\tau)}{\theta_1(\frac{\alpha}{N}|\tau)} \right|^2 . \end{aligned} \tag{4.15}$$

Orthogonality of momentum eigenstates was used here. The eigenvalues of the g^α on $\mathbf{k}$ are obviously $\mathrm{e}^{\pm 2\pi \mathrm{i}\nu}$, with $\nu = \alpha/N$. As such, the trigonometric factors $2\sin\pi\nu$ are seen to be absorbed by the momentum integration.

In the twisted sectors there is no momentum, so that for $\beta \neq 0$ we simply find

$$\mathcal{Z}(g^\beta, g^\alpha) \equiv Z_{\text{n.c.}}(g^\beta, g^\alpha) = |\chi_{g^\beta}^{(X) g^\alpha}|^2 \tag{4.16}$$

(see appendix B for explicit expressions of the twisted chiral blocks $\chi_{g^\beta}^{(X) g^\alpha}$).

The modular generators S and T are thus seen to act on the $Z_{\text{n.c.}}(g^\beta, g^\alpha)$, for all β, α, by an appropriate exchange of the boundary conditions (see eq. (4.11)). Therefore, the partition function corresponding to eq. (4.1),

$$Z_{\text{n.c.}} = \frac{1}{N} \sum_{\beta,\alpha=0}^{N-1} Z_{\text{n.c.}}(g^\beta, g^\alpha) = \frac{1}{N} \left\{ Z_{\text{flat}} + \sum_{\alpha=1}^{N-1} Z_{\text{n.c.}}(e, g^\alpha) \right\} + \mathcal{Z}^{\mathbb{Z}_N}_{\text{twisted}} \,, \tag{4.17}$$

is modular invariant. The notation used here is such that $\mathcal{Z}^{\mathbb{Z}_N}_{\rm twisted}$ is the contribution of twisted sectors from a single $\mathbb{Z}_N$ fixed point:

$$\mathcal{Z}^{\mathbb{Z}_N}_{\rm twisted} = \frac{1}{N}\sum_{\beta=0}^{N-1}\sum_{\alpha=1}^{N-1} |\chi^{g^\beta}_{g^\alpha}|^2 \,, \tag{4.18}$$

where g generates $\mathbb{Z}_N$.

Next consider the compact case, where $\mathbb{C}$ is compactified on a $\mathbb{Z}_N$ invariant lattice Λ. Geometrically, the only consistent choices, giving rise to a quotient space with isolated orbifold singularities, are $N = 2, 3, 4, 6$.

Momenta take values in the dual lattice. Furthermore, there are now closed string winding states. The unprojected, untwisted partition function $Z_{\rm torus}$ that is now obtained is again modular invariant by itself. The untwisted contribution performing the projection onto invariant states is now found to be

$$\begin{aligned}\mathcal{Z}(e, g^\alpha) &= \sum_{\mathbf{n},\mathbf{w}} \langle \mathbf{n};\mathbf{w}| g^\alpha \, q^{\frac{\alpha'}{2}\frac{\mathbf{n}^2}{R^2} + \frac{(\mathbf{w}R)^2}{2\alpha'}} |\mathbf{n};\mathbf{w}\rangle \, |\widehat{\chi}^{(X)\,g^\alpha}_e|^2 = |\widehat{\chi}^{(X)\,g^\alpha}_e|^2 \,, \\ &= 4\sin^2\frac{\pi\alpha}{N}\, Z_{\rm n.c.}(e, g^\alpha) \,, \end{aligned} \tag{4.19}$$

since only $|\mathbf{n} = 0; \mathbf{w} = 0\rangle$ is invariant under the action of g. Contrary to the non-compact case, the S modular transform of the untwisted terms is that of the chiral blocks $\chi^{(X)\,g^\alpha}_e$ in eq. (4.12). More precisely, we have

$$\mathcal{Z}(e, g^\alpha) \xrightarrow{S} 4\sin^2\left(\frac{\pi\alpha}{N}\right) \mathcal{Z}(g^\alpha, e) \,. \tag{4.20}$$

Adding all sectors with weight one like in eq. (4.17) would no longer yield a modular invariant. Eq. 4.20 requires that the terms $(1/N)\sum_{\beta=1}^{N-1} 4\sin^2(\pi\beta/N) \;\; \mathcal{Z}(g^\beta, e)$ be added to the untwisted projected contribution $\frac{1}{N}(Z_{\rm torus} + \sum_{\alpha=1}^{N-1} \mathcal{Z}(e, g^\alpha))$. To ensure full modular invariance, the remaining sectors must be included with appropriate weigths determined by these.

It is only in the cases $N = 2, 3, 4, 6$ that the factors $N_\beta = 4\sin^2(\pi\beta/N)$ are all integers. These are precisely the cases where a geometrical interpretation of the quotient space is possible. More precisely, N_β counts the number of points on T^2 fixed under the action of the element g^β of the orbifold group; this is basically the content of the Lefschetz fixed-point theorem. As such, the modular invariant partition function fixed in terms of the factors N_α in eq. (4.20) incorporates the contributions of twisted sectors from all the points fixed under (subgroups of) the orbifold group. In other words, it has exactly the structure of eq. (4.6).

Let us illustrate this in the concrete example of $T^2/\mathbb{Z}_6$. There are several fixed points now. There is one $\mathbb{Z}_6$-fixed point A (fixed by the generator g) in figure 1. Further, there are two $\mathbb{Z}_3$-fixed points a and b in figure 1, (fixed by g^2). These points are combined in a *single* orbit under the $\mathbb{Z}_6$ action of g. Finally, there are 3 $\mathbb{Z}_2$-fixed

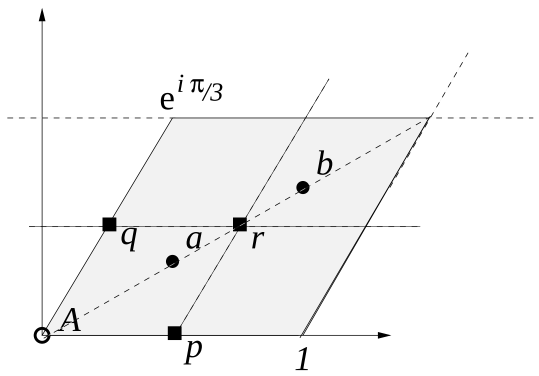

Figure 1: Fixed points for $T^2/\mathbb{Z}_6$. See the text for details.

points (p,q and r), fixed by g^3; they form a *single* orbit under the $\mathbb{Z}_6$ action. The modular invariant partition function is

$$Z = \frac{1}{6}\Big\{ Z_{\rm torus} + \sum_{\alpha=1}^{5} \mathcal{Z}(e, g^\alpha) + N_1 \sum_{\alpha=0}^{5} \mathcal{Z}(g, g^\alpha) + N_2 \sum_{\alpha=0}^{2} \mathcal{Z}(g^2, g^{2\alpha}) +$$
$$+ N_3 \sum_{\alpha=0}^{1} \mathcal{Z}(g^3, g^{3\alpha}) + N_4 \sum_{\alpha=0}^{3} \mathcal{Z}(g^4, g^{2\alpha}) + N_5 \sum_{\alpha=0}^{5} \mathcal{Z}(g^5, g^\alpha) +$$
$$+ N_1 \sum_{\alpha=1,3,5} \mathcal{Z}(g^2, g^\alpha) + N_1 \sum_{\alpha=1,2,4,5} \mathcal{Z}(g^3, g^\alpha) + N_1 \sum_{\alpha=1,3,5} \mathcal{Z}(g^4, g^\alpha) \Big\}\,, \tag{4.21}$$

with $N_1 = N_5 = 4\sin^2(\pi/6) = 1$, and analogously $N_2 = N_4 = 3$ and $N_3 = 4$. In this expression, all possible twisted contributions at points fixed by elements g^β appear correctly counted with the fixed point multiplicities $N_\beta = N_{6-\beta}$. This partition function is rewritten more suggestively in the form of eq. (4.1):

$$Z = \frac{1}{6}\left\{ Z_{\rm torus} + \sum_{\alpha=1}^{5} \mathcal{Z}(e, g^\alpha) \right\} + \mathcal{Z}^{\mathbb{Z}_6}_{\rm twisted} + \mathcal{Z}^{\mathbb{Z}_3}_{\rm twisted} + \mathcal{Z}^{\mathbb{Z}_2}_{\rm twisted}\,, \tag{4.22}$$

using the notation introduced in eq. (4.18). The $\mathbb{Z}_3$ and $\mathbb{Z}_2$ twisted contributions appear only once, as we are now identifying the fixed points belonging to the same orbits.

4.3 Boundary states for fractional branes

Here and below, we will restrict our attention to D-branes that are pointlike along the orbifold directions. The orbifold space is taken to be $\mathbb{C}^n/\Gamma$; following [54] a natural starting point are Γ-invariant configurations of D-branes on the covering space $\mathbb{C}^n$.

As such the brane and its images naturally give rise to the regular representation[6] $\mathcal{R}$ on the Chan-Paton factors. Correspondingly, these D-branes have been called

[6]The regular representation is the representation of dimension $|\Gamma|$ obtained regarding the group products as linear operations on a vector space spanned by the group elements themselves. Namely, the matrix $\mathcal{R}(g)$ representing g has matrix elements $[\mathcal{R}(g)]_{g_1 g_2} = \delta_{g_2, g\cdot g_1}$.

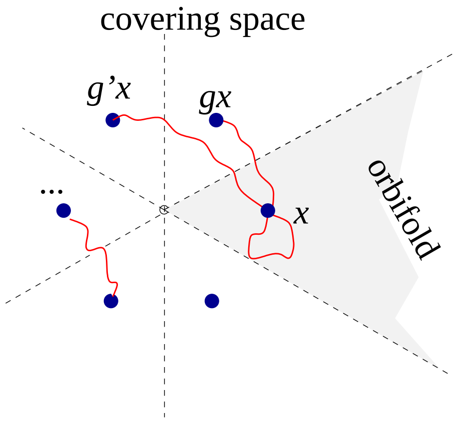

Figure 2: A Γ-invariant configuration of D-branes (blue dots) in the covering spaces. Open strings carry Chan-paton indices in the regular representation $\mathcal{R}$.

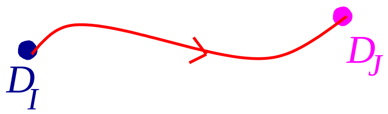

Figure 3: Open strings between fractional branes of types I and J carry Chan-Paton labels in the irreducible representations $\bar{\mathcal{D}}_I$ and $\mathcal{D}_J$.

regular branes. A particularly well-studied example is given by regular D-branes on $\mathbb{C}^2/\Gamma$, where the low-energy world-volume theory yields a supersymmetric σ-model on the ALE-space resolving the singularity.[7]

However, the regular representation of a discrete group Γ is not irreducible. It decomposes as

$$\mathcal{R} = \bigoplus_I d_I \, \mathcal{D}^I \,, \tag{4.23}$$

in terms of the irreducible representations $\mathcal{D}^I$ of Γ, whose dimensions we denote by d_I. Thus one may wonder whether there exists a more "elementary" set of D-branes such that the open strings attached to the latter carry Chan-Paton indices transforming in an irreducible representation. This turns out to be the case, and such D-branes have been called *fractional* D-branes [49, 7, 50, 8] (see also [57] for a review).

A fractional Dp-brane is a BPS object. It carries only a fraction of the charge with respect to the untwisted RR $(p+1)$–form of a usual Dp-brane but, contrarily to a usual brane, it is charged with respect to some twisted RR $(p+1)$–form [7]. It is stuck at the orbifold fixed point (where all twisted fields sit): indeed, to place it elsewhere, we should associate it to an invariant configuration in in the covering space. However, as already discussed (see figure 2), such an invariant configuration corresponds to the regular representation.

Fractional branes can be geometrically interpreted as higher-dimensional branes wrapped on exceptional cycles of the resolved space [7, 8]. The exceptional cycles collapse in the orbifold limit, but a non-zero flux of B on them persists [5], leaving us

[7] This has been rephrased as follows [54, 55]: the matter fields of the world–volume theory give a physical realisation of the mathematical hyperkähler quotient construction of ALE spaces, originally due to Kronheimer [56].

with lower-dimensional fractional branes with non-zero mass. The determination of the spectrum of D-branes in the theory should thus be linked with the determination of the homology of the (resolved) orbifold space or, more precisely, to its homological K-theory [58, 22, 23, 34].

As already mentioned, we concentrate solely on the fractional branes. Our aim is to derive the corresponding consistent boundary states by adapting Cardy's construction.[8] This procedure applies equally well to abelian and non-abelian orbifolds, and it links very explicitly (for instance in the case of $\mathbb{C}^2/\Gamma$) the boundary states that represent the fractional p-branes at the orbifold point with the geometrical picture of such branes as higher-dimensional branes wrapped on vanishing cycles.

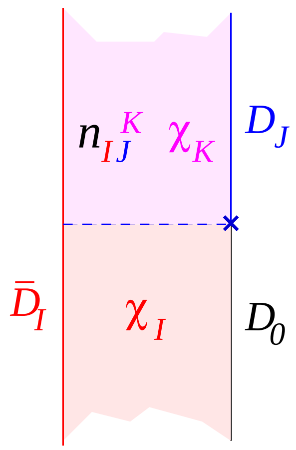

Figure 4: The boundary conditions corresponding to fractional branes fuse in correspondence to the algebra of irreducible representations of Γ.

We concentrate first on describing the boundary states at the level of the CFT that describes the $\mathbb{C}^n/\Gamma$ orbifold space. In terms of these it is straightforward to write the boundary states of the complete string theory under consideration, be it a bosonic or a supersymmetric string theory.

The cylinder amplitudes with fixed boundary conditions of type I and J are described, from the point of view of open strings, as Γ-projected one-loop traces

$$Z_{IJ}(q) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \mathrm{Tr}_{IJ} \left(\hat{g}\, q^{L_0 - c/24} \right) , \tag{4.24}$$

where $\hat{g}$ acts not only on the string fields X^μ and ψ^μ, but also on the Chan-Paton labels at the two open string endpoints, transforming respectively in the representation $\bar{\mathcal{D}}_I$ and $\mathcal{D}_J$. The amplitude Z_{IJ} is basically constructed out of the untwisted chiral blocks $\chi_e^g(q)$ discussed in section 4.2. Indeed, performing the Chan-Paton part of the traces, we have

$$\begin{aligned} Z_{IJ}(q) &= \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \mathrm{tr}_{\bar{\mathcal{D}}_I}(g)\, \mathrm{tr}_{\mathcal{D}_J}(g)\, \mathrm{Tr} \left(g\, q^{L_0 - c24} \right) \\ &= \frac{1}{|\Gamma|} \sum_{\alpha} n_\alpha\, (\rho_I^\alpha)^* \rho_J^\alpha\, \chi_e^{g^{(\alpha)}}(q) \, . \end{aligned} \tag{4.25}$$

Here and below the index α will label the conjugacy classes $\mathcal{C}^\alpha$ (containing n_α elements) of Γ and the group element $g^{(\alpha)}$ is a representative of $\mathcal{C}^\alpha$; moreover, ρ_J^α denotes the character matrix, namely $\rho_J^\alpha = \mathrm{tr}_{\mathcal{D}_J}(g^{(\alpha)})$.

[8] The boundary states for fractional branes have already been considered in the literature [22, 24, 25]. Their derivation via Cardy's procedure, however, will allow us to clarify several fine points.

The consistent amplitudes Z_{IJ} can be expanded with integer multiplicities on a basis of chiral blocks, $\chi_I(q)$:

$$Z_{IJ}(q) = \sum_K n_{IJ}^K \, \chi_K(q) \,. \tag{4.26}$$

The chiral blocks χ_I are amplitudes in which only the Chan-Paton representation $\mathcal{D}_I$ runs in the loop. They correspond to amplitudes Z_{0I}, where at one endpoint the Chan-Paton labels transform in the trivial identical representation $\mathcal{D}_0$. It follows from eq. (4.25) that these blocks are simply the discrete Fourier transforms of the chiral blocks χ_e^g:

$$\chi_I(q) = \frac{1}{|\Gamma|} \sum_\alpha n_\alpha \, \rho_I^\alpha \, \chi_e^{g^{(\alpha)}}(q) \,. \tag{4.27}$$

To carry out Cardy's construction, we need the S-modular transformation properties of the open string basis $\chi_I(q)$ of chiral blocks. Via eq. (4.27), these are determined by the transformations of the untwisted chiral blocks χ_e^g.

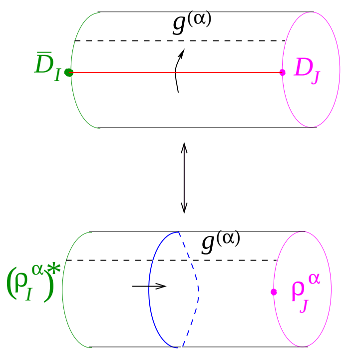

Figure 5: The open string one loop trace with b.c.s. of type I and J with the insertion of an element $g^{(\alpha)}$ (top) corresponds upon S-modular and discrete Fourier transform to the propagation in the tree channel of a closed string twisted by $g^{(\alpha)}$ (bottom). ρ denotes the character matrix.

Bosonic theory. Consider first the purely bosonic case. Recall that we consider open strings with Dirichlet boundary conditions in all of the orbifold directions, so that there is no momentum zero-mode. The S modular transformation of the relevant chiral blocks $\chi_e^g = \hat{\chi}_h^{(X)g}$ of eq. (4.9) reads

$$\chi_e^g \xrightarrow{S} \sigma(e,g) \, \chi_g^e \,, \tag{4.28}$$

where the factors $\sigma(e,g)$ are as follows (see eq. (4.12)):

$$\begin{aligned} \sigma(e,e) &= 1 \,, \\ \sigma(e,g) &= \prod_{l=1}^{n} (2 \sin \pi \nu_{g,l}) \,, \qquad g \neq e \,, \end{aligned} \tag{4.29}$$

if we are considering a $\mathbb{C}^n/\Gamma$ orbifold.

From eq. (4.27) and eq. (4.28) it follows that the χ_I blocks, when transformed to the tree channel, are expressed them in terms of twisted chiral blocks. Twisted blocks are labeled by the conjugacy classes of Γ; it is convenient to introduce the simplified notation $\chi_\alpha(\tilde{q})$ for a basis of the tree-channel blocks:

$$\chi_\alpha(\tilde{q}) = \chi_{g^{(\alpha)}}^e(\tilde{q}) \,, \qquad g^{(\alpha)} \in \mathcal{C}^\alpha \,. \tag{4.30}$$

We can then write the modular transformation as

$$\chi_I(q) = \sum_\alpha \mathcal{S}_I^\alpha \, \chi_\alpha(\tilde{q}) \,, \tag{4.31}$$

with

$$\mathcal{S}_I^\alpha = \frac{n_\alpha\,\rho_I^\alpha}{|\Gamma|}\,\sigma(e,g^{(\alpha)})\,. \tag{4.32}$$

Notice that the entries of $\mathcal{S}_0^\alpha$, namely $\sigma(e,g^{(\alpha)})\,n_\alpha/|\Gamma|$, are all positive, as appropriate to the distinguished role of the identity representation in Cardy's procedure. The fusion algebra that follows from applying Verlinde's formula to the matrix $\mathcal{S}$,

$$n_{IJ}^K = \sum_\alpha \frac{\mathcal{S}_I^\alpha \mathcal{S}_J^\alpha (\mathcal{S}^{-1})_\alpha^K}{\mathcal{S}_0^\alpha}\,, \tag{4.33}$$

corresponds simply[9] to the algebra of the irreducible representations of Γ:

$$\mathcal{D}_I \otimes \mathcal{D}_I = \sum_K n_{IJ}^K\,\mathcal{D}_K\,. \tag{4.35}$$

In terms of this matrix $\mathcal{S}$ we can now proceed with Cardy's construction of consistent boundary states, as in section 2.3.

"Ishibashi states" $|g^{(\alpha)}\rangle\rangle$ solve the analogue of the gluing condition eq. (A.3) (of Dirichlet type) in a sector twisted by $g^{(\alpha)}$ (see [26]):

$$|g^{(\alpha)}\rangle\rangle = \exp\left(\sum_{l=1}^{n}\left[\sum_{\kappa_l}\frac{\bar{\alpha}^l_{-\kappa_l}\tilde{\alpha}^l_{-\kappa_l}}{\kappa_l} + \sum_{\bar{\kappa}_l}\frac{\alpha^l_{-\bar{\kappa}_l}\tilde{\bar{\alpha}}^l_{-\bar{\kappa}_l}}{\bar{\kappa}_l}\right]\right)|0,g^{(\alpha)}\rangle\,. \tag{4.36}$$

The modings of the oscillators $\alpha^l_{\kappa_i}$ follows from the twisted identifications eq. (4.8), and were implicitly used in the computations of the twisted chiral blocks: we have $\kappa_l \in \mathbb{Z}+\nu_{\alpha,l}$, while $\bar{\kappa}_l \in \mathbb{Z}-\nu_{\alpha,l}$, where $\exp(2\pi i\nu_{\alpha,l})$ is the eigenvalue of $g^{(\alpha)} \in \mathcal{C}^\alpha$ on the complex field X^l. By $|0,g^{(\alpha)}\rangle$ we have denoted the vacuum in the sector twisted by $g^{(\alpha)}$. All the directions in the orbifold space are uncompactified Dirichlet directions; if in the l-th complex direction the sector defined by $g^{(\alpha)}$ appears as untwisted (i.e., if $\nu_{\alpha,l}=0$), there is a zero-mode, and the boundary state includes a zero-mode part of the type $\delta^2(\hat{X}^l)|k^l=0\rangle$, as discussed in section A.

In particular, the untwisted part of the boundary state is the only one that emits closed strings that carry momentum in all of the orbifold directions; as we shall see, also the normalization of the untwisted component of the boundary state in the full string theory differs from the one of the twisted components.

In fact, the "Ishibashi states" $|g^{(\alpha)}\rangle\rangle$ are not quite what we want: for one thing, in the case of a non-abelian Γ they are not invariant under the orbifold projection.

---

[9] The inverse of the matrix $\mathcal{S}$ of eq. (4.30) is given by $(\mathcal{S}^{-1})_\alpha^J = (\rho_J^\alpha)^*/\sigma(e,g^{(\alpha)})$, as it follows from the orthogonality relations of the characters. Then in the Verlinde formula eq. (4.31) all factors $\sigma(e,g^{(\alpha)})$ cancel and one gets

$$n_{IJ}^K = \frac{1}{|\Gamma|}\sum_\alpha n_a\,\rho_I^\alpha\,\rho_J^\alpha\,(\rho_k^a)^*\,, \tag{4.34}$$

which is also the expression that follows from eq. (4.35), using again the orthogonality properties of characters.

Rather, they mix with "Ishibashi states" in sectors twisted by conjugate group elements. The invariant Ishibashi state associated to a conjugacy class $\mathcal{C}^\alpha$ reads

$$|\alpha\rangle\rangle = \frac{1}{\sqrt{n_\alpha}} \sum_{g^{(\alpha)} \in \mathcal{C}^\alpha} |g^{(\alpha)}\rangle\rangle \,. \tag{4.37}$$

The states $|\alpha\rangle\rangle$ are orthogonal, satisfying:

$$\langle\langle\alpha|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-c/12)}|\beta\rangle\rangle = \delta_{\alpha\beta}\,\chi_\alpha(\tilde{q})\,. \tag{4.38}$$

The consistent boundary state associated to a boundary condition of type I (i.e., with Chan-Paton labels in the representation $\mathcal{D}_I$) is then obtained via Cardy's formula eq. (2.17) and can be written as

$$\begin{aligned} |I\rangle &= \frac{1}{\sqrt{|\Gamma|}} \sum_\alpha \sqrt{n_\alpha\,\sigma(e,g^{(\alpha)})}\,\rho_I^\alpha\,|\alpha\rangle\rangle \\ &= \sum_\alpha \psi_I^\alpha \sqrt{\sigma(e,g^{(\alpha)})}\,|\alpha\rangle\rangle\,, \end{aligned} \tag{4.39}$$

where we introduced the quantities

$$\psi_I^\alpha = \sqrt{\frac{n_\alpha}{|\Gamma|}}\,\rho_I^\alpha\,. \tag{4.40}$$

In terms of the boundary states $|I\rangle$, the open-string one-loop amplitude $Z_{IJ}(q)$, transformed to the tree channel, is simply

$$Z_{IJ}(\tilde{q}) = \langle I|\tilde{q}^{\frac{1}{2}(L_0+\tilde{L}_0-c/12)}|J\rangle\,. \tag{4.41}$$

(Super)string theory. All the expressions obtained so far must be supplemented with the contributions from the directions transverse to the orbifold space. Furthermore, the appropriate normalisations must be taken into account in order to get the full-fledged physical string boundary states that give the closed string tree-channel description of the cylinder open string *amplitude*. The only subtlety in completing this programme is common to both the bosonic and the superstring, and resides in a proper treatment of *closed* string bosonic zero-modes (momenta) along orbifold directions.

The one-loop open string amplitude $\mathcal{Z}_{IJ}$ corresponding to boundary conditions I and J in the orbifold CFT and with $p+1$ world-volume directions (transverse to the orbifold) is given by (3.3) with the flat cylinder partition function $Z_{1,2}^{(d)}(q)$ replaced by $Z_{1,2}^{(d-2n)}(q)\,Z_{IJ}(q)$, where Z_{IJ} was given in eq. (4.24). In the case of the superstring, the fermionic sectors, discussed in section 3.2 for the flat space part, and in next paragraph for the orbifold CFT, must be taken into account appropriately.

Discrete Fourier and modular transformations turn the open string amplitude $\mathcal{Z}_{IJ}$ into the amplitude for closed string propagation between boundary states $|B,\alpha\rangle\rangle$, exactly as in eq. (3.4). Apart from the orbifold Ishibashi states $|\alpha\rangle\rangle$, the full Ishibashi states $|B,\alpha\rangle\rangle$ also contain flat space components of section A (and section 3.2) accounting for the transverse directions. Moreover, they have to be properly normalised so as to ensure the equality of the amplitude in the two channels. These normalisations are readily obtained following the discussion in [13] that leads to the usual normalisation N_p of eq. (3.5). One sees that the more general result in our situation is

$$N_p^{(\alpha)} = \frac{\sqrt{\pi}}{2}\, 2^{\frac{10-d^{(\alpha)}}{4}} \left(2\pi\sqrt{\alpha'}\right)^{\frac{d^{(\alpha)}-4}{2}-p}, \tag{4.42}$$

where $d^{(\alpha)}$ is the number of Neumann directions $(p+1)$ plus the number of Dirichlet directions in which the $\delta(\hat{x})|k=0\rangle$ zero-mode part (see the final remarks in appendix A) is present. Thus, in the untwisted case we have $d^{(0)} = d$ and we get eq. (3.5), while in the twisted sectors[10] $\alpha \neq 0$ we have $d^{(\alpha)} = d - 2n$. The charges in the twisted sectors are thus given by the analogues of eq. (3.26), namely by

$$\mu_p^{(\alpha)} = 2\sqrt{2}\, 2^{-n/2}\, N_p^{(\alpha)} = \sqrt{2\pi}\left(2\pi\sqrt{\alpha'}\right)^{3-n-p} \tag{4.43}$$

for $d = 10$. Eq. (4.42) agrees with the results of [24].

The *untwisted* fields (such as the graviton and dilaton and the untwisted RR forms) emitted by the Dp-brane at the orbifold fixed point propagate in all directions, including those along the orbifold. To have canonically normalised bulk kinetic terms for these fields, which depend on all the coordinates of $\mathbb{R}^{1,d-2n-1} \times \mathbb{C}^n/\Gamma$, we define them as $\sqrt{|\Gamma|}$ times the fields that are canonically normalised on the covering space $\mathbb{R}^{1,d-1}$. As a consequence, the tension and charge normalisations $T_p^{(\Gamma)}$ and $\mu_p^{(\Gamma)}$ with respect to the canonically normalised fields in $\mathbb{R}^{1,d-2n-1} \times \mathbb{C}^n/\Gamma$ are related to the flat space expressions given in eq. (3.5) and eq. (3.26) respectively; in particular,

$$T_p^{(\Gamma)} = \frac{T_p}{\sqrt{|\Gamma|}}, \qquad \mu_p^{(\Gamma)} = \frac{\mu_p}{\sqrt{|\Gamma|}}. \tag{4.44}$$

No such redefinition is required for the twisted fields, since these do not propagate in the orbifold. Notice that a further factor of $1/\sqrt{|\Gamma|}$ is incorporated in the orbifold states $|I\rangle$ (both twisted and untwisted) via the definition eq. (4.40) of the ψ_I^α coefficients.

Fermionic sectors. In the superstring case, the transformation of the fermionic parts adds no complication beyond those already discussed for flat space. The relevant chiral blocks are now products of the bosonic and fermionic ones given in section 4.2 and appendix B. Let us denote them by

$$(\chi_a)_h^g = \hat{\chi}_h^{(X)g}\, (\chi_a^{(\psi)})_h^g, \tag{4.45}$$

[10] Assuming for simplicity that in the twisted sector no zero-modes are present in any of the complex directions along the orbifold, see the discussion after eq. (4.36).

JHEP01(2001)038

with $a = v, o, s, c$. Then the S-modular transformation of an untwisted chiral block reads

$$(\chi_a)^g_e \xrightarrow{S} \sigma(e, g) \sum_m (S_{(2n)})_a^m (\chi_m)^e_g\,, \tag{4.46}$$

where $S_{(2n)}$ is the modular matrix for the fermionic characters of SO$(2n)$ (if we are considering a $\mathbb{C}^n/\Gamma$ orbifold), given in eq. (3.7), and the factors $\sigma(e, g)$ are again those of eq. (4.29). We see that the modular transformation consists in a direct product of the transformation acting on the SO$(2n)$ labels v, o, s, c and on the Γ-labels. Thus one can straightforwardly proceed to define a basis of open-string chiral blocks $\chi_{I,a}(q)$ (with $a = v, o, s, c$) and one of tree-level ones $\chi_{\alpha,m}(q')$ (with $m = v, o, s, c$), related by the modular transformation

$$\chi_{I,a}(q) = \sum_\alpha \sum_m \mathcal{S}_I^\alpha\, (S_{(2n)})_a^m\, \chi_{\alpha,m}(q')\,. \tag{4.47}$$

As far as boundary states are concerned, one starts, analogously to the flat space case, with states

$$|\alpha;\sigma;\eta\rangle\rangle = |\alpha\rangle\rangle\, |\alpha;\sigma;\eta\rangle\rangle_\psi\,, \tag{4.48}$$

with $\sigma = \mathrm{NS}, \mathrm{R}$ and $\eta = \pm$. In eq. (4.48) the bosonic states $|\alpha\rangle\rangle$ of eq. (4.36) are multiplied by fermionic ones $|\alpha;\sigma;\eta\rangle\rangle_\psi$, which solve the overlap condition for the (complex) fermionic oscillators:

$$\psi^l_{r_l} = \mathrm{i}\eta\, \tilde{\psi}^l_{-r_l}\,, \tag{4.49}$$

($l = 1, \ldots, n$), where the modings r_l are dictated by the $g^{(\alpha)}$ twist and the R or NS sector: $r_l \in \mathbb{Z} + \nu_{\alpha,l}$ in the R sector, and an extra shift of 1/2 in the NS sector.[11] Then a proper orthogonal basis of Ishibashi states $|\alpha, m\rangle\rangle$ ($m = v, o, s, c$) is defined by taking combinations as in eqs. (3.13)–(3.16). Finally, consistent Cardy states (for instance, in type 0 theory) $|I, a\rangle$, with $a = v, o, s, c$, are given by

$$|I, a\rangle = \sum_\alpha \sum_m \frac{\mathcal{S}_I^\alpha\, (S_{(2n)})_a^m}{\sqrt{\mathcal{S}_0^\alpha\, (S_{(2n)})_v^m}}\, |\alpha, m\rangle\rangle\,. \tag{4.50}$$

The fusion algebra for the boundaries (I, a) is the direct product of the algebras of the irreducible representations of Γ (with structure constants n_{IJ}^K) and of the (modified) algebra of the v, o, s, c classes of SO$(2n)$ described in eqs. (3.19)–(3.20).

Analogously, one constructs the consistent states in a type-II theory, according to the discussion in section 3.2. It is not difficult to show that the consistent string boundary states constructed according to the prescription we just described agree with the expressions proposed in [25] of the boundary states of fractional branes for (non-abelian) orbifolds; the salient coefficients $\psi_I^\alpha \sqrt{\sigma(e, g^{(\alpha)})}$ are already recognised in the bosonic expressions eq. (4.39).

[11] The explicit solution to the condition eq. (4.49) is in terms of a coherent-state-like expression analogous to the bosonic one, eq. (4.36), see, e.g., [26]. Moreover, when fermionic zero-modes are present, extra contributions analogous to the ones in eq. (3.11) must be taken into account.

Intersection index. In [59], following earlier suggestions in [60, 61], the relevance of the of the Witten index $\mathcal{I}_{IJ}$ for the open strings stretched between two branes I and J was pointed out. This index is the one-loop amplitude with boundary conditions of type I and J in the $\mathrm{R}(-)^F$ sector:

$$\mathcal{I}_{IJ} \equiv Z_{IJ}^{\mathrm{R}(-)^F} = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \mathrm{Tr}_{IJ;\mathrm{R}} \left(\hat{g}\, (-)^F\, q^{L_0 - c/24} \right) . \tag{4.51}$$

It is a topological quantity that counts the unpaired chiral massless fermion states living on the intersection of the two D-branes. This is the CFT counterpart of the geometrical intersection form between cycles; more precisely, when the D-branes in question admit a geometrical interpretation as wrapped higher-dimensional branes, it coincides with the intersection form of the cycles on which they are wrapped.

From the cancellation of the non-zero mode contributions to eq. (4.51), the latter is q-independent, hence topological. This is a generic feature of $\mathrm{R}(-)^F$ chiral blocks. In particular,

$$\hat{\chi}_e^{(X)g^{(\alpha)}}(q)\, \chi_{0,e}^{0,g^{(\alpha)}}(q) = \prod_{l=1}^{n} (-2\mathrm{i}\, \sin \pi \nu_{\alpha,l}) \,, \tag{4.52}$$

as follows from eq. (4.9) and eq. (B.12) (the fermionic characters in the $\mathrm{R}(-)^F$ sector, $\chi_{0,e}^{0,g^{(\alpha)}}(q)$, are defined in appendix B). Thus we get (see eq. (4.25))

$$\mathcal{I}_{IJ} = (-\mathrm{i})^n \sum_{\alpha} (\psi_I^\alpha)^* \, \psi_J^\alpha \prod_{l=1}^{n} (2 \sin \pi \nu_{\alpha,l}) \,. \tag{4.53}$$

In the tree channel, this amplitude corresponds to ($(-\mathrm{i})^n$ times) the RR contribution (odd spin structure contribution); it is given by

$$\mathcal{I}_{IJ} = (-\mathrm{i})^n \, \langle\langle I; R; \pm | \tilde{q}^{\frac{1}{2}(L_0 + \tilde{L}_0 - c/12)} | J; R; \pm \rangle\rangle \,, \tag{4.54}$$

where we provisionally introduced the states[12]

$$|J; R; \pm\rangle\rangle = \sum_{\beta} \psi_J^\beta \sqrt{\sigma(e, g^{(\beta)})}\, |\beta; R; \pm\rangle\rangle \,. \tag{4.56}$$

This gives the same result as in eq. (4.53), as for the odd spin structure we have

$$\langle\langle \alpha; R; \pm | \tilde{q}^{\frac{1}{2}(L_0 + \tilde{L}_0 - c/12)} | \beta; R; \pm \rangle\rangle = c_\alpha \, \delta_{\alpha\beta} \,, \tag{4.57}$$

[12] The tree-channel expression eq. (4.54) can of course also be expressed in terms of the consistent boundary states of eq. (4.50), e.g.,

$$\mathcal{I}_{IJ} = (\langle I, s| - \langle I, c|) \tilde{q}^{\frac{1}{2}(L_0 + \tilde{L}_0 - c/12)} (|J, v\rangle - |J, o\rangle) \,, \tag{4.55}$$

but this way of writing obscures somehow the fact that it is only the odd spin structure that is being considered.

with $c_\alpha = 1$ if $\alpha \neq 0$ and $c_0 = 0$, where the index 0 labels the conjugacy class of the identity; this is due to the presence of zero-modes in the untwisted case.

If we were to compute the full string amplitude in the $\mathrm{R}(-)^F$ sector, the amplitude $\mathcal{I}_{IJ}$ (like all the amplitudes we have been discussing in this section) would be multiplied by the contributions from the directions transverse to the orbifold. The full string amplitude thus vanishes if any Ramond zero-modes are present in these directions. Nevertheless, when a geometrical interpretation of the fractional branes is possible, $\mathcal{I}_{IJ}$ maintains its interest as describing the intersection form of exceptional cycles.

4.3.1 Branes on ALE spaces and the McKay correspondence

Let us next focus attention on complex two-dimensional orbifolds $\mathbb{C}^2/\Gamma$, with Γ a discrete subgroup of SU(2). Thus, elements $g \in \Gamma$ act of the complex coordinates (X^1, X^2) in the defining 2-dimensional representation $\mathcal{Q}$:

$$g : \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \to \mathcal{Q}(g) \cdot \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}, \tag{4.58}$$

with $\mathcal{Q}(g) \in \mathrm{SU}(2)$. Resolving the orbifold singular point gives rise in this case to asymptotically locally euclidean (ALE) spaces,[13] whose boundary at infinity is S^3/Γ. The discrete subgroups of SU(2) are in one-to-one relation to the simply-laced ADE Lie algebras. McKay [39] described this correspondence explicitly as follows (see for instance [63] for a mathematical review of the classical McKay correspondence and its generalisations to higher dimensions). The tensor product of the defining 2-dimensional representation $\mathcal{Q}$ of Γ with an irreducible representation $\mathcal{D}_I$ decomposes as

$$\mathcal{Q} \otimes \mathcal{D}_I = \oplus_J \widehat{A}_I{}^J \, \mathcal{D}_J \,. \tag{4.59}$$

It was observed that the $r \times r$ matrix $\widehat{A}$ (where r is the number of irreducible representations of Γ) coincides with the adjacency matrix of the extended Dynkin diagram corresponding to a simply-laced algebra[14] $\mathcal{G}_\Gamma$ of rank $r-1$. The extended Cartan matrix of the Lie algebra is given by

$$\widehat{C}_I{}^J = 2\delta_I{}^J - \widehat{A}_I{}^J \,. \tag{4.60}$$

Take an element $g^{(\alpha)}$ in the $\mathcal{Q}$ representation, and diagonalise it:

$$\mathrm{diag}(\mathrm{e}^{2\pi \mathrm{i}\nu_\alpha}, \mathrm{e}^{-2\pi \mathrm{i}\nu_\alpha}) \,.$$

[13] See for instance [62, 54, 55] for reviews on ALE spaces.

[14] In particular, the cyclic Z_k groups are related to the A_{k-1} Dynkin diagrams, the binary extended dihedral groups $\mathbb{D}_k$ to the D_{k+2} diagrams and the binary extensions of the tetrahedron, octahedron and icosahedron groups to, respectively, E_6, E_7 and E_8.

The character of $\mathcal{Q}$ evaluated on $g^{(\alpha)}$ is thus $2\cos 2\pi\nu_\alpha = 2 - 4\sin^2\pi\nu_\alpha$. Taking traces and using the orthogonality of characters, one then derives from McKay's relation eq. (4.59) that

$$\widehat{C}_I{}^J = \sum_\alpha \psi_I^\alpha \, (\psi_J^\alpha)^* \, 4\sin^2\pi\nu_\alpha \,. \tag{4.61}$$

The quantities ψ_I^α introduced in eq. (4.40) are the eigenvectors of the extended Cartan matrix $\widehat{C}_I{}^J$, and eigenvalues of the latter are in correspondence with conjugacy classes of Γ; the eigenvalues are $\sigma_\alpha = 4\sin^2\pi\nu_\alpha$.

As is well known from the work of Kronheimer [56], the Dynkin diagram of the Lie algebra $\mathcal{G}_\Gamma$ plays a fundamental geometrical role. Resolving the singularity of the $\mathbb{C}^2/\Gamma$ space gives rise to a smooth ALE space $\mathcal{M}_\Gamma$, which has a non-trivial middle homology group $H_2(\mathcal{M}_\Gamma, \mathbb{Z})$ of dimension $r-1$, the rank of $\mathcal{G}_\Gamma$. The (symmetric) intersection matrix of the $r-1$ exceptional two-cycles c_i $(i = 1, \ldots, r-1)$, is given by the non-extended Cartan matrix $C_i{}^j$ of $\mathcal{G}_\Gamma$. Thus the 2-cycles $c_i \in H_2(\mathcal{M}_\Gamma, \mathbb{Z})$ correspond to the simple roots α_i of $\mathcal{G}_\Gamma$. Furthermore, the cycle $c_0 = -\sum_i d_i c_i$ is associated to the negative of the highest root, $\alpha_0 = -\sum_i d_i \alpha_i$, i.e., to the extra node in the *extended* Dynkin diagram. Here d_i are the Coxeter numbers and $d_0 = 1$; in the McKay relation the d_I's correspond to the dimensions of the irreducible representations of Γ. The intersection matrix of the cycles $\{c_I\} = \{c_0, c_i\}$ is then the negative of the *extended* Cartan matrix $\widehat{C}_I{}^J$.

Fractional branes as wrapped branes. The fractional Dp-branes of type $|I\rangle$ correspond to D$(p+2)$-branes wrapped on an *integer* basis of cycles $c_I \in H_2(\mathcal{M}_\Gamma, \mathbb{Z})$, in the collapsing limit in which these cycles shrink.[15]

The first test of this interpretation is provided by the the topological partition function [24, 25] $\mathcal{I}_{IJ} = Z_{IJ}^{\mathrm{R}(-)^F}$ between Dp-branes of types I and J, that computes the intersection of the cycles c_I and c_J. In the case of $\mathbb{C}^2/\Gamma$, eq. (4.54) gives

$$\mathcal{I}_{IJ} = -\sum_\alpha \psi_I^\alpha \, (\psi_J^\alpha)^* \, 4\sin^2\pi\nu_\alpha = -\widehat{C}_{IJ} = c_I \cdot c_J \,, \tag{4.62}$$

in full correspondence with the above discussion of the non-trivial 2-cycles of $\mathcal{M}_\Gamma$.

The geometrical interpretation of the fractional branes as wrapped branes must furthermore be consistent with their masses and charges (both twisted and untwisted).

The boundary states for the consistent fractional branes encode the corresponding D-brane couplings to the RR $(p+1)$-form fields $A_{p+1}^{(\alpha)}$ that arise in the untwisted $(\alpha = 0)$ and twisted $(\alpha \neq 0)$ sectors of type-II string theory on $\mathbb{R}^{1,5} \times \mathbb{C}^2/\Gamma$. The RR

[15] The supergravity classical solution corresponding to a fractional D-brane can be retrieved [64] in accordance with this interpretation.

components of the boundary states in the orbifold CFT read explicitly

$$|I;R:\pm\rangle = \psi_I^0 |0;R;\pm\rangle\rangle + \sum_{\alpha\neq 0} \psi_I^\alpha\, 2\sin\pi\nu_\alpha |\alpha;R;\pm\rangle\rangle\,. \tag{4.63}$$

Let us focus first on the branes of type $|i\rangle$, i.e., those corresponding to non-trivial representations of Γ. The charges of such branes with respect to the twisted fields $A_{p+1}^{(\alpha)}$, with $\alpha \neq 0$, are encoded in the matrix

$$Q_i{}^\alpha = \mu_p^{(\alpha)}\, \psi_i{}^\alpha \sqrt{\sigma^\alpha}\,, \qquad (i,\alpha = 1,\dots,r-1)\,, \tag{4.64}$$

where $\sigma^\alpha = 4\sin^2\pi\nu_\alpha$ are the non-zero eigenvalues of the *extended* Cartan matrix; we have included the overall normalisation $\mu_p^{(\alpha)}$ of (4.43).

Considering the theory on the resolved ALE space, the geometrical counterparts of the twisted gauge fields [7] are the Kaluza-Klein $p+1$-forms $\mathcal{A}_{p+1}^{(i)}$ that come from the harmonic decomposition

$$A_{p+3} = \sum_i \mathcal{A}_{p+1}^{(i)} \wedge \omega_i\,. \tag{4.65}$$

Here, A_{p+3} is the ten-dimensional A_{p+3} RR form and the ω_i's are the normalisable anti-self-dual $(1,1)$-forms, Poincaré dual to the cycles c_i, so that

$$\int_{\mathcal{M}_\Gamma} \omega_i \wedge \omega_j = \int_{c_i} \omega_j = c_i \cdot c_j = -C_{ij}\,. \tag{4.66}$$

Notice that both the $\mathcal{A}_{p+1}^{(i)}$ and the $A_{p+1}^{(\alpha)}$ fields depend only on the six coordinates transverse to the orbifold, the first ones because of their KK origin, the second ones because of the absence of momentum along the orbifold directions in the twisted sectors.

A D$(p+2)$-brane wrapped on the cycle c_i, which corresponds to the fractional brane $|i\rangle$ in the orbifold limit, is charged under the fields $\mathcal{A}_{p+1}^{(j)}$ because of its Wess-Zumino coupling

$$\mu_{p+2} \int_{\mathrm{D}(p+2)} A_{p+3} = \mu_{p+2} \int_{\mathrm{D}p} \mathcal{A}_{p+1}^{(j)} \int_{c_i} \omega_j = -C_{ij}\, \mu_p^{(\alpha)} \int_{\mathrm{D}p} \mathcal{A}_{p+1}^{(j)}\,. \tag{4.67}$$

Notice that the usual D$(p+2)$ charge μ_{p+2}, which is appropriate for the branes in the smooth resolved space $\mathbb{R}^{1,5} \times \mathcal{M}_\Gamma$, equals the twisted charge normalisation $\mu_p^{(\alpha)}$, as follows from eqs. (3.26), (4.43). Comparing the charge matrices $Q_{i\alpha}$ of eq. (4.64) to $\mu_p^{(\alpha)} C_{ij}$ encoding the charges with respect to the "geometric" fields $\mathcal{A}^{(j)}$, one is led to conclude that the two bases of $(p+1)$-forms are related by

$$\mathcal{A}_{p+1}^{(i)} = -\sum_{j,\alpha} (C^{-1})^{ij} \psi_j{}^\alpha \sqrt{\sigma^\alpha} A_{p+1}^{(\alpha)}\,. \tag{4.68}$$

In a matrix notation we write this change of basis as $\mathcal{A}_{p+1} = C^{-1}\psi^T\sqrt{\Sigma}A_{p+1}$. This relation is consistent with the bulk kinetic terms for these fields. Indeed, from a canonical kinetic term for A_{p+3}, under the decomposition eq. (4.67) and making use of eq. (4.66), we get a six-dimensional bulk kinetic term

$$\frac{1}{2}\int_{\mathbb{R}^{1,5}} \bar{\mathcal{F}}^{(i)}_{p+2} \wedge C_{ij}\, {}^*\mathcal{F}^{(j)}_{p+2} = \frac{1}{2}\int_{\mathbb{R}^{1,5}} \bar{F}^{(\alpha)}_{p+2} \wedge {}^*F^{(\alpha)}_{p+2}\,, \tag{4.69}$$

where in the second step we changed basis according to eq. (4.68). The fact that the kinetic terms become canonical in terms of the twisted fields $A^{(\alpha)}_{p+1}$ is an non-trivial consistency requirement: closed string twisted sectors are mutually orthogonal and as such cannot produce a non-diagonal effective kinetic term. The fact that the change of basis eq. (4.68) indeed performs this diagonalisation is not obvious a priori. A proof involves the fact that (with $\alpha,\beta\neq 0$)

$$\sum_{i,j}(\psi^*)_\alpha^{\ i}\,(C^{-1})_i^{\ j}\,\psi_j^{\ \beta} = \frac{1}{\sigma^\alpha}\,\delta^\alpha_{\ \beta}\,. \tag{4.70}$$

This property can be proven using that the ψ^α_I are eigenvectors of the *extended* Cartan matrix[16] with corresponding orthogonality properties

$$\sum_{I=0}^{r-1}(\psi^\alpha_{\ I})^*\psi_I^{\ \beta} = \delta^{\alpha\beta}\,, \qquad \sum_{\alpha=0}^{r-1}(\psi_I^{\ \alpha})^*\psi^\beta_{\ J} = \delta_{IJ}\,. \tag{4.71}$$

In fact, the latter follow from the properties of the character matrix ρ (see appendix C.2), in view of $\psi^\alpha_I = \sqrt{n^\alpha/|\Gamma|}\,\rho^\alpha_I$.

The twisted charges of the fractional brane $|0\rangle$ associated to the trivial representation are described by the vector $Q^\alpha = \mu_p^{(\alpha)}\,\psi_0{}^\alpha\sqrt{\sigma^\alpha} = -\mu_p^{(\alpha)}\sqrt{\sigma^\alpha/|\Gamma|}$. They are correctly accounted for by assuming that this state corresponds to a brane wrapped on the cycle $c_0 = -\sum d_i c_i$, as is suggested by McKay's correspondence and easily shown using again the orthogonality properties eq. (4.71).

The fractional branes $|I\rangle$ are furthermore charged with respect to the RR untwisted $(p+1)$-form A_{p+1}. It follows from eq. (4.63) and from the discussion around eq. (4.44) that their untwisted charges are

$$Q_I = \mu_p^{(\Gamma)}\,\frac{d_I}{\sqrt{|\Gamma|}} = \mu_p\,\frac{d_I}{|\Gamma|}\,. \tag{4.72}$$

[16]Contrary to one's naive first guess, the closed string basis $A^{(\alpha)}_{p+1}$ is not related to the "geometrical" one $\mathcal{A}^{(i)}_{p+1}$ by the change of basis $\mathcal{A}_{p+1} = q^T\Lambda^{-1/2}A_{p+1}$, where q is the (orthonormal) matrix of eigenvectors of the Cartan matrix and Λ the diagonal matrix containing its eigenvalues. Although this transformation makes the kinetic terms canonical, the true closed-string basis requires a further transformation $U = \Lambda^{-1/2}\,q\,\psi^T W\Sigma^{1/2}$. Since the latter is unitary the canonical form obtained by the first transformation is preserved.

From the geometric point of view, the untwisted charges Q_I arise from the Wess-Zumino coupling

$$\mu_{p+2}\int_{\mathrm{D}(p+2)} A_{p+1}\wedge(2\pi\alpha'\mathcal{F}+B)=\mu_{p+2}\int_{\mathrm{D}p} A_{p+1}\int_{c_I}(2\pi\alpha'\mathcal{F}+B)\,, \tag{4.73}$$

since the orbifold conformal field theory is obtained with a non-trivial B-background [5, 7, 6, 65] localised on the vanishing cycles. Consider indeed the branes of type $|i\rangle$. The charges Q_i of (4.72) are retrieved from (4.73) by wrapping a brane with no world-volume gauge field $\mathcal{F}$ and assuming

$$B=-\frac{\mu_p}{\mu_{p+2}}\frac{1}{|\Gamma|}\sum_i d_i(C^{-1})^{ij}\omega_j=\frac{\mu_p}{\mu_{p+2}}\frac{1}{|\Gamma|}\sum_i d_i\omega^i\,. \tag{4.74}$$

The (1,1)-forms $\omega^i=-(C^{-1})^{ij}\omega_j$ satisfy the property that $\int_{c_i}\omega^j=\delta^i_j$. Notice also that $\mu_p/\mu_{p+2}=4\pi^2\alpha'$. The expression of B given above agrees with the one obtained in [65, sect. 4] as a local limit from the requirement of consistent embeddings of orbifold CFT's into the part of moduli space of $\mathcal{N}=(4,4)$ theories describing K3 sigma-models.

Consider however the brane of type $|0\rangle$. To get from the Wess-Zumino term (4.73) the untwisted charge $Q_0=\mu_p/|\Gamma|$, in addition to having the B-background (4.74) we must be wrapping a D$(p+2)$ brane with a non-trivial open string $\mathcal{F}$ field [7, 66, 67]. Indeed, the B-contribution from wrapping over the cycle c_0 would be given by $-\mu_p\sum_i d_i^2/|\Gamma|=\mu_p(1-|\Gamma|)/|\Gamma|$. To cancel the extra negative charge we need a $\mathcal{F}$ background localised on c_0. Such a background is quantised, as its Chern class is $\int_{c_0}\mathcal{F}=2\pi k$, with $k\in\mathbb{Z}$. In our case, we just have to take $k=1$ to get from (4.73) the sought for contribution of $4\pi^2\alpha'\mu_{p+2}=\mu_p$.

From the above discussion it follows that a superposition of fractional branes that transforms in the regular representation, namely $\sum_I d_I|I\rangle$ according to the decomposition eq. (4.23), is not charged under the twisted sector gauge fields and has untwisted charge $\mu_p\sum_I(d_I)^2/|\Gamma|=\mu_p$. Thus it is degenerate with a bulk Dp-brane located at the orbifold fixed point, as expected.

Relation to branes on ALE spaces at their Gepner points. In the present subsection we have been studying D-branes on a $\mathbb{C}^2/\Gamma$ orbifold. Geometrically, such orbifolds are degenerate ALE spaces, the two-cycles c_i having zero size. However, the orbifold CFT is non-singular due to a non-zero B-flux through these vanishing cycles [5, 6]. In the recent papers [51, 52], D-branes have been studied at a different point of the ALE moduli space, the Gepner point. The CFT in question may be formulated as a coset model based on

$$\left(\frac{\mathrm{SU}(2)_{h-2}}{\mathrm{U}(1)}\times\frac{\mathrm{SL}(2)_{h+2}}{\mathrm{U}(1)}\right)\Big/\mathbb{Z}_h\,, \tag{4.75}$$

where h is the Coxeter number of $\mathcal{G}_\Gamma$. By constructing boundary states and computing their intersection index, the interpretation in terms of branes wrapping two-cycles was derived. The fact that the intersection index of [51, 52] agrees with the one of the orbifold theory described here is no surprise: given the fact that string theory on an ALE space preserves sixteen supersymmetries, one expects the spectrum of BPS D-branes to be the same in both theories.

5. Discrete torsion

In [37], it was noted that it is consistent with modular invariance to include certain phases in the orbifold partition function $Z(\tau, \bar{\tau})$ on a torus with modular parameter τ:

$$Z = \frac{1}{|G|} \sum_{\substack{g,h \in G \\ [g,h]=e}} Z(g,h)\varepsilon(g|h)\,. \tag{5.1}$$

The phases $\varepsilon(g|h)$ are called discrete torsion.

Higher loop modular invariance implies the following consistency conditions on $\varepsilon(g|h)$ [37]:

$$\varepsilon(gh|k) = \varepsilon(g|k)\varepsilon(h|k) \tag{5.2}$$

$$\varepsilon(g|h) = \varepsilon(h|g)^{-1} \tag{5.3}$$

$$\varepsilon(g|g) = 1\,. \tag{5.4}$$

From the hamiltonian (rather than the functional integral) point of view, $\varepsilon(g|h)$ changes the phase of the operator corresponding to g in the sector twisted by h. States in $\mathcal{H}_h$ that were invariant under the original action of N_h, will only survive the orbifold projection with discrete torsion if $\varepsilon(g|h) = 1$ for all $g \in N_h$.

D-branes in theories with discrete torsion were introduced in [38], at least for abelian orbifold groups. We shall review this construction, generalised to non-abelian G,[17] and construct boundary states for D-branes localised at an orbifold fixed point.

To study D-branes in an orbifold theory, one starts with a G-symmetric configuration of D-branes on the covering space of the orbifold. In addition to the space-time action $r(g)$, one considers a representation $\gamma(g)$ of G in the gauge group of the D-branes. The fields ϕ living on the D-branes are then projected as

$$\gamma^{-1}(g)\phi\gamma(g) = r(g)\phi\,. \tag{5.5}$$

In theories without discrete torsion, γ is a genuine representation of the orbifold group. In theories with discrete torsion, γ is taken to be a projective representation [38]:

$$\gamma(g)\gamma(h) = \omega(g|h)\gamma(gh)\,. \tag{5.6}$$

[17] Recently, this generalization has also been considered in [35].

As we will discuss soon, the *factor system* $\omega(g|h)$, which consists of non-zero complex numbers, is related to the discrete torsion phases $\varepsilon(g|h)$. Before we come to that, we briefly study factor systems by themselves.

From

$$\gamma(ghk) = \omega(gh|k)^{-1}\gamma(gh)\gamma(k) = \omega(gh|k)^{-1}\omega(g|h)^{-1}\gamma(g)\gamma(h)\gamma(k) \tag{5.7}$$

and

$$\gamma(ghk) = \omega(g|hk)^{-1}\gamma(g)\gamma(hk) = \omega(g|hk)^{-1}\omega(h|k)^{-1}\gamma(g)\gamma(h)\gamma(k)\,, \tag{5.8}$$

we deduce the *associativity condition*

$$\omega(gh|k)\omega(g|h) = \omega(g|hk)\omega(h|k)\,. \tag{5.9}$$

Rescaling the representation matrices by $\gamma(g) = \beta(g)\gamma'(g)$, we find the *equivalent factor system*

$$\omega'(g|h) = \frac{\beta(gh)}{\beta(g)\beta(h)}\omega(g|h)\,. \tag{5.10}$$

More abstractly, the associativity condition eq. (5.9) and the equivalence relation eq. (5.10) are combined in the statement that the factor system is a two-cocycle in the group cohomology $H^2(G, \mathrm{U}(1))$.

Consider two commuting elements $g, h \in G$. From the factor system ω we define

$$\varepsilon(g|h) = \omega(g|h)\omega(h|g)^{-1}\,, \qquad [g,h] = e\,. \tag{5.11}$$

As the notation suggests, $\varepsilon(g|h)$ will be identified with a discrete torsion phase. We now check that this identification could make sense. Doing these checks, we shall extensively use the fact that $[g,h] = e$. Note, however, that $\varepsilon(g|h)$ is only defined for commuting group elements, so that our discussion is in no way restricted to abelian orbifold groups. First, taking determinants in eq. (5.6) and the same equation with g and h reversed, we find that $|\varepsilon(g|h)| = 1$. Second, it is easy to check that eq. (5.11) is invariant under the equivalence relation eq. (5.10). Third, eq. (5.3) and eq. (5.4) are manifestly satisfied. Finally, eq. (5.2) follows from eq. (5.9).

Of course, these consistency conditions alone do not imply that eq. (5.11) makes sense. For a certain abelian orbifold, eq. (5.11) was derived in [38] by considering a specific genus one diagram with one boundary. In what follows, we shall check the relation eq. (5.11) by factorising a cylinder amplitude between two D-branes in the closed string channel. We take the D-branes to be localised at a fixed point of the orbifold.[18] We shall show that from $\mathcal{H}_h$ only states invariant under the N_h projection *with* $\varepsilon(g|h)$ *included* contribute to the amplitude. The factorization in

[18]For D-branes extended in the orbifold directions, the relation between discrete torsion and projective representations is more subtle [33].

the closed string channel will be achieved by constructing boundary states for the D-branes with discrete torsion. It will then be sufficient to check that these boundary states are well-projected.

The boundary states for fractional branes in the theory without discrete torsion were given in eq. (4.50). The Ishibashi components are weighted by factors proportional to the character ρ_I^α, where α denotes the twisted sector to which the Ishibashi component belongs and I is the representation of the orbifold group associated to the fractional brane. The consistent boundary states in a theory with discrete torsion are obtained by replacing ρ_I^α by a projective character $\rho(g^{(\alpha)})$ of a group element in the conjugacy class $\mathcal{C}^\alpha$.[19] This raises the following question. The Ishibashi states eq. (4.37) were invariant under the orbifold action *without* discrete torsion. When discrete torsion is turned on, some of these Ishibashi states will no longer be physical. How do the boundary states we have just constructed know about that? To answer this question, we have to study projective characters in a bit more detail.

The character ρ associated to a projective representation γ is defined as

$$\rho(h) = \operatorname{tr}\gamma(h)\,. \tag{5.12}$$

Unlike the character of genuine representations, the character of a projective representation is not a class function. In general, we have

$$\rho(ghg^{-1}) = \omega(gh|g^{-1})^{-1}\omega(g|h)^{-1}\omega(g^{-1}|g)\omega(h|e)\rho(h)\,. \tag{5.13}$$

If g and h commute, the left hand side of eq. (5.13) reduces to $\rho(h)$, so $\rho(h)$ can only be non-zero if the product of ω's on the right hand side of eq. (5.13) equals one. Because g and h commute, we can use associativity to write

$$\omega(gh|g^{-1})^{-1} = \omega(h|g)\omega(h,e)^{-1}\omega(g,g^{-1})^{-1}\,, \tag{5.14}$$

such that the product of ω's on the right hand side of eq. (5.13) becomes just

$$\omega(gh|g^{-1})^{-1}\omega(g|h)^{-1}\omega(g^{-1}|g)\omega(h|e) = \varepsilon(h|g)\varepsilon(g^{-1}|g) = \varepsilon(h|g)\,. \tag{5.15}$$

Here, ε is as defined in eq. (5.11) and the fact that $\varepsilon(g^{-1}|g) = 1$ follows easily from eq. (5.2) and eq. (5.4). Thus, we can conclude that if a $g \in N_h$ exists such that $\varepsilon(g|h) \neq 1$, then

$$\rho(h) = 0\,. \tag{5.16}$$

For our boundary states with discrete torsion, this property implies that Ishibashi states that are not physical in the theory with discrete torsion do not contribute to the consistent boundary states. This is a direct consistency check for the relation between discrete torsion and projective representations for D-branes at an orbifold fixed point.

[19] As we are about to discuss, projective characters are not class functions, so the freedom in choosing $g^{(\alpha)} \in \mathcal{C}^\alpha$ leads to an ambiguity. Since the matrices acting on Chan-Paton factors have determinants of modulus one, the ambiguity is just a phase factor, which is irrelevant for our purposes.

6. Conclusions

In this paper we have shown how to generalise Cardy's construction of consistent boundary states in the case of complex orbifolds. The distinctive feature is that the operation taking Ishibashi states (cohomology) into Cardy states (homology) involves a generalised discrete Fourier transform. The latter naturally implements the known identification of fractional D-branes at the fixed point with irreducible representations of the orbifold group. In particular, we put quite some effort in the proper identification of the twisted RR form basis that is natural from the geometric point of view, i.e., the basis obtained after KK reduction on the Poincaré duals of the vanishing cycles. As such, the constructed boundary states were shown explicitly to allow an interpretation as branes wrapping cycles with some particular B-flux. Although attention was mainly focused on the case of complex orbifold surfaces, it is obvious that the Cardy construction also applies in more involved cases.

In the final section we moved attention to orbifolds with discrete torsion. The boundary states describing pointlike branes on these orbifolds paved the way to establish the link between D-branes as projective representations on the one hand and discrete torsion phases in the closed string partition function on the other hand.

Acknowledgments

We are grateful to D. Berenstein, P. De Smet, M. Frau, M. Gaberdiel, L. Gallot, J. Harvey, D. Kutasov, A. Lerda, E. Martinec, I. Pesando, R. Russo and S. Sethi for interesting discussions. We would like to thank the University of Torino (B.C.) and the organisers of the "Semestre Cordes" at the Centre Emile Borel of the Institut Henri Poincaré (F.R.) for hospitality during the initial and final stages of this work, respectively. This work was supported by DOE grant DE-FG02-90ER-40560, NSF grant PHY-9901194, by the European Commission TMR programme ERBFMRX-CT96-0045 and by the RTN programme HPRN-CT-2000-00131. F.R. was supported in part by a Marie Curie Fellowship of the European Community Programme "Improving human research potential and the socio-economic knowledge base" under contract number HPMT-CT-2000-00163. During part of this work, B.C. was Aspirant FWO-Flanders.

A. Boson boundary state

This appendix reviews the construction of consistent boundary states in the CFT of free bosons. [41, 42]

Consider first a single boson on a circle of radius R. The sub-algebra $\mathcal{A}$ is then generated by ∂X, and contains the identity operator. In the holomorphic and anti-holomorphic sectors, the Fourier modes α_n and $\tilde{\alpha}_n$ are defined by

$$\partial X(z) = -\mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \alpha_m z^{-m-1}\,; \qquad \bar{\partial} X(\bar{z}) = -\mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \tilde{\alpha}_m \bar{z}^{-m-1} \tag{A.1}$$

and obey the algebra $[\alpha_m, \alpha_n] = [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\, \delta_{m+n}$ On a generic circle this CFT has an infinite number of highest weight states $|(k,w)\rangle$ $(k, w \in \mathbb{Z})$. These have no oscillators excited and are thus defined by

$$\begin{aligned} \hat{p}|(k,w)\rangle &= \frac{k}{R}|(k,w)\rangle \\ \hat{w}|(k,w)\rangle &= w|(k,w)\rangle\,; \end{aligned} \tag{A.2}$$

with $\hat{p} = (\alpha_0 + \tilde{\alpha}_0)/\sqrt{2\alpha'}$, and $\hat{w} = \sqrt{\alpha'/2}\,(\alpha_0 - \tilde{\alpha}_0)/R$.

Taking an automorphism Ω of $\mathcal{A}_R$, one may impose the following gluing conditions at world-sheet boundaries (see eq. (2.6)):

$$(\alpha_n + \Omega(\tilde{\alpha}_{-n}))\, |i\rangle\rangle_\Omega = 0\,. \tag{A.3}$$

On $|i\rangle\rangle_\Omega$ the left-moving and right-moving closed string Virasoro operators get identified: $(L_0^c - \tilde{L}_0^c)|i\rangle\rangle_\Omega = 0$.

The special case in which Ω equals the identity corresponds to Neumann boundary conditions, while Dirichlet boundary conditions are realised by $\Omega(\bar{\partial}X) = -\bar{\partial}X$. The generalised Ishibashi states solving eq. (A.3) for the special cases of Dirichlet or Neumann gluing conditions will be denoted by $|i\rangle\rangle_D, |i\rangle\rangle_N$, respectively.

With Neumann boundary conditions, the Ishibashi states are

$$|(0,w)\rangle\rangle_N = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n}\tilde{\alpha}_{-n}\right) |(0,w)\rangle \tag{A.4}$$

with corresponding Ishibashi characters eq. (2.11)

$$\chi_{N,w}(\tilde{q}) = \frac{\tilde{q}^{R^2 w^2/4\alpha'}}{\eta(\tilde{q})}\,. \tag{A.5}$$

The Ishibashi states of eq. (A.4) satisfy the orthogonality condition eq. (2.12). Notice that the highest weight representations j of the chiral preseved chiral algebra are labeled by the winding number $j = (0, w)$ here. The orthonormal set of basis vectors $|j; N\rangle$ becomes $|(0,w); \{m_n\}\rangle$ in this concrete setting, where m_n denote the α_{-n} oscillator numbers.

As to the boundary state/D-brane correspondence, one has to specify which open strings/boundary conditions are physically sensible. With Neumann boundary conditions along the compact X direction, one allows open strings to carry non-zero momentum along that direction, so that the corresponding open string character

$\chi(q)_N = \mathrm{Tr}_N(q^{L_0^o})$ contains a momentum sum. Moreover, one can turn on a Wilson line $A_X = \theta/2\pi R$ with $\theta \in [0, 2\pi[$; this is equivalent to shifting the open string momenta: $n/R \to n/R - \theta/2\pi R$. Thus the generic open string character $\chi(q)_{N,\theta}$ reads

$$\begin{aligned}\chi(q)_{N,\theta} &= \sum_{n\in\mathbb{Z}} \frac{q^{(n/R-\theta/2\pi R)^2}}{\eta(q)} \\ &= \frac{R}{\sqrt{2\alpha'}} \sum_{w\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\theta w} \frac{\tilde{q}^{R^2 w^2/4\alpha'}}{\eta(\tilde{q})} = \frac{R}{\sqrt{2\alpha'}} \sum_{w\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\theta w} \chi_{N,w}(\tilde{q})\,, \end{aligned} \tag{A.6}$$

where in the second line we performed a Poisson resummation. The corresponding consistent boundary states are

$$|\theta\rangle_N = \left(\frac{R}{\sqrt{2\alpha'}}\right)^{1/2} \sum_{w\in\mathbb{Z}} e^{\mathrm{i}\theta w} |(0,w)\rangle\rangle_N\,. \tag{A.7}$$

Notice that these states take the form of eq. (2.17), with $S_\theta^w = \frac{R}{\sqrt{2\alpha'}}\, e^{\mathrm{i}\theta w}$. The role of the distinguished Cardy's state $|\mathbf{0}\rangle$ is clearly played by the state corresponding to no Wilson line, $|\mathbf{0}\rangle_N$.

Turning to Dirichlet boundary conditions, we take $\Omega(\bar{\partial}X) = -\bar{\partial}X$. This is evidently an implementation of T-duality (which is a one-sided parity transformation). Instead of eq. (A.4) one now has generalised Ishibashi states

$$|(k,0)\rangle\rangle_D = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n}\tilde{\alpha}_{-n}\right) |(k,0)\rangle \tag{A.8}$$

solving eq. (A.3) with corresponding Ishibashi characters eq. (2.11)

$$\chi_{D,k} = \frac{\tilde{q}^{\alpha' k^2/4R^2}}{\eta(\tilde{q})}\,. \tag{A.9}$$

D-branes localised along the X direction make sense, and the corresponding boundary states are constructed as follows. Since the Dirichlet boundary condition at both ends sets the open string momentum along X to zero but allows for non-zero winding, which must accordingly be summed over in the open string character. For an open string stretched between two D-branes at a distance Δx from each other, the character $\mathrm{Tr}_D(q^{L_0^o})$ is

$$\begin{aligned}\chi(q)_{D,\Delta x} &= \frac{q^{(2\pi R w + \Delta x)^2/4\pi^2\alpha'}}{\eta(q)} \\ &= \frac{\sqrt{\alpha'}}{\sqrt{2}R} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\Delta x k/R} \frac{\tilde{q}^{\alpha' k^2/4R^2}}{\eta(\tilde{q})} = \frac{\sqrt{\alpha'}}{\sqrt{2}R} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\Delta x k/R} \chi_{D,k}(\tilde{q})\,, \end{aligned} \tag{A.10}$$

where in the second line we went to the tree channed by Poisson resummation. The corresponding consistent boundary states describe D-branes localised at fixed positions x along the circle:

$$|x\rangle_D = \left(\frac{\sqrt{\alpha'}}{\sqrt{2}R}\right)^{1/2} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}xk/R} |(k,0)\rangle\rangle_D \,. \tag{A.11}$$

Also in this case, these state assumes the form of eq. (2.17). The distinguished state $|\mathbf{0}\rangle$ is of course simply $|x=0\rangle_D$. The close analogy between eq. (A.7) and eq. (A.11) reflects the fact that turning on Wilson lines results in shifting positions of branes in the T-dual picture.

The above discussion was restricted to the case of one free boson so as to simplify the discussion. As a nice result, one verifies that the ratio of the coefficients in front of $|\theta=0\rangle_N$ and $|x=0\rangle_D$ is $(2\pi R)/2\pi\sqrt{\alpha'}$ thus confirming that the relative tensions differ by $1/2\pi\sqrt{\alpha'}$ [19].

Moreover, regarding the states in eq. (A.7) and eq. (A.11) as constituents, the generalisation to the case of several bosons is obvious and consists of taking tensor products of the appropriate constituents.

To study the decompactification limit $R\to\infty$, note that the states $|(k,0)\rangle$ appearing in eq. (A.8) are normalised to one. Thus they differ from the states $e^{\mathrm{i}k\hat{x}}|0\rangle$ that are usually considered in string theory by a factor of $1/\sqrt{2\pi R}$. Taking into account the $1/\sqrt{R}$ factor manifest in eq. (A.11), the R-dependence will be just right to turn the discrete momentum sum in eq. (A.11) into a momentum integral. This momentum integral then gives the usual delta function localising the brane in the transverse directions (see, for instance, [43, 44]).

B. Chiral blocks

In this appendix we collect the explicit expressions for the chiral blocks of orbifolds of free bosonic and fermionic theories. As discussed at the beginning of section 4.2, we consider orbifolds of $\mathbb{C}^n$ by a point group Γ, and in each sector (g,h) we can diagonalise simultaneously both h and g, as in eq. (4.7) and eq. (4.8). As a result, one can consider separately each complex direction. We can therefore limit ourselves to describe here the chiral blocks

$$\chi^g_h(q) = \mathrm{Tr}_{\mathcal{H}_h}\, g\, q^{L_0-c/24} \tag{B.1}$$

for a single complex bosonic field X and a fermionic one ψ, for which a sector denoted by (g,h) is characterised by a twist by h which amounts to

$$X\in\mathcal{H}_h: \;\; X(\tau,\sigma+2\pi) = \mathrm{e}^{2\pi\mathrm{i}\nu} X(\tau,\sigma) \tag{B.2}$$

and by an action of g given by

$$g:\ X \mapsto \mathrm{e}^{2\pi \mathrm{i}\nu'} X \tag{B.3}$$

(and similarly for ψ). Of course, $\bar{X}$ and $\bar{\psi}$ transform with the opposite phases.

Bosonic characters. For a complex boson X in a twisted sector we get

$$\begin{aligned}
\chi_h^{(X)\,g}(q) &= q^{-1/12+\frac{1}{2}\nu'(1-\nu')} \prod_{n=0}^{\infty} \left(1-\mathrm{e}^{2\pi\mathrm{i}\nu} q^{n+\nu'}\right)^{-1} \left(1-\mathrm{e}^{-2\pi\mathrm{i}\nu} q^{n+(1-\nu')}\right)^{-1} \\
&= \mathrm{i}\,\mathrm{e}^{-\pi\mathrm{i}\nu} q^{-\nu'^2/2} \frac{\eta(\tau)}{\theta_1(\nu+\tau\nu'|\tau)}\,.
\end{aligned} \tag{B.4}$$

In the first line we used the expression of the zero-point energy of a complex boson in the sector twisted by h:

$$a^{(X)} = \frac{1}{12} - \frac{1}{2}\nu'(1-\nu')\,, \tag{B.5}$$

and in the second step we used eq. (C.10). Notice that when writing infinite products in terms of θ functions, ν and ν' will always assume values in the range $[0,1[$.

In the untwisted sector ($h=e$) bosonic zero modes show up. In the closed string torus compactification, for instance, they are the momentum and winding modes. These modes, however, are shared between the chiral and anti-chiral sectors and there is no natural split. Their expression and their effect is discussed, by means of an explicit example, in section 4.2. Here we concentrate on the non-zero-mode contribution which, for a generic insertion g, reads

$$\widehat{\chi}_e^{(X)\,g}(q) = q^{-1/12} \prod_{n=1}^{\infty} \left(1-\mathrm{e}^{2\pi\mathrm{i}\nu} q^{n}\right)^{-1} \left(1-\mathrm{e}^{-2\pi\mathrm{i}\nu} q^{n}\right)^{-1} = 2\sin\pi\nu\,\frac{\eta(\tau)}{\theta_1(\nu|\tau)}\,. \tag{B.6}$$

The character in the untwisted sector with no insertions coincides with the flat space expression. The resulting partition function is of the form

$$Z(e,e) = Z_{\mathrm{flat}} \propto \mathrm{Vol}\,\frac{1}{\mathrm{Im}\,\tau\,|\eta(\tau)|^2}\,, \tag{B.7}$$

where $1/|\eta(\tau)|^2$ is the contribution of chiral and anti-chiral non-zero modes while $1/\mathrm{Im}(\tau)$ results from the gaussian zero-mode integration (which is replaced by a sum over the appropriate dual lattice in the compactified case). The flat space partition function is modular invariant by itself.

Fermionic characters. Consider now a complex fermion ψ (and its complex conjugate). Let us first introduce some more notation, to distinguish the various fermion characters. The R sector will be denoted by $\zeta'=0$, while the NS sector corresponds to $\zeta' = 1/2$; that is, ζ' corresponds to the shift in the modings of the oscillators appropriate to the two sectors. Another label $\zeta = 0(1/2)$ will denote a Fock space trace with (without) insertion of $(-)^F$. The insertion of the fermion number $(-)^F$ in the trace is denoted by $\zeta = 0$, while $\zeta = 1/2$ denotes the trace without the fermion number insertion. As an example, $\chi^{0,g}_{1/2,h}$ is the character in the NS$(-)^F$ sector, twisted by h and with the insertion of g.

The zero-point energies in the R and NS sectors can be written compactly as

$$a^{(\zeta',\nu')} = \frac{1}{24} - \frac{1}{2}\left(\tilde{\nu}' + \zeta' - \frac{1}{2}\right)^2 , \tag{B.8}$$

where $\tilde{\nu}'$ is defined an integer shift[20] of ν' such that $\tilde{\nu}' + \zeta' < 1$. Thus, in the R sector we get $a^{(\mathrm{R})} = -a^{(X)} = -1/12 + \nu'(1-\nu')/2$, while in the NS sector we have to distinguish between $\nu' < 1/2$ and $\nu' > 1/2$. Taking into account eq. (B.8), the chiral blocks for free fermions are

$$\chi^{\zeta,g}_{\zeta',h}(q) = q^{\tilde{\nu}'^2/2}\, \mathrm{e}^{-2\pi \mathrm{i}(\zeta'-1/2)(\nu+\zeta-1/2)}\, \frac{\theta\left[{1-2\zeta' \atop 1-2\zeta}\right](\tau + \tilde{\nu}'\tau|\tau)}{\eta(\tau)} . \tag{B.9}$$

For instance, in the R$(-)^F$ sector, we have

$$\begin{aligned} \chi^{0,g}_{0,h}(q) &= q^{\frac{1}{12} - \frac{\nu'}{2}(1-\nu')} \prod_{n=0}^{\infty} (1 - \mathrm{e}^{2\pi \mathrm{i}\nu} q^{n+\nu'})(1 - \mathrm{e}^{-2\pi \mathrm{i}\nu} q^{n+1-\nu'}) \\ &= -\mathrm{i}\mathrm{e}^{\pi \mathrm{i}\nu}\, \frac{\theta\left[{1 \atop 1}\right](\nu + \tau\nu'|\tau)}{\eta(\tau)} . \end{aligned} \tag{B.10}$$

One may notice that the expression of NS and NS$(-)^F$ characters twisted by ν' coincides (upon use of some theta function identities) with the one of R and R$(-)^F$ characters twisted by $\tilde{\nu}' + 1/2$.

Instead of the above set of characters, it is often more useful to use a different set, labeled by the o, v, s, c irreducible representations of the SO(2) algebra. Taking the relevant combinations of the characters in eq. (B.9), one finds

$$\begin{aligned} (\chi_o)^g_h &= q^{(\tilde{\nu}')^2/2}\, \frac{\theta_3(\nu + \tau\tilde{\nu}'|\tau) + \theta_4(\nu + \tau\tilde{\nu}'|\tau)}{2\eta(\tau)} , \\ (\chi_v)^g_h &= q^{(\tilde{\nu}')^2/2}\, \frac{\theta_3(\nu + \tau\tilde{\nu}'|\tau) - \theta_4(\nu + \tau\tilde{\nu}'|\tau)}{2\eta(\tau)} , \\ (\chi_s)^g_h &= q^{(\nu')^2/2}\, \mathrm{e}^{\mathrm{i}\pi\nu}\, \frac{\theta_2(\nu + \tau\nu'|\tau) - \mathrm{i}\theta_1(\nu + \tau\nu'|\tau)}{2\eta(\tau)} , \\ (\chi_c)^g_h &= q^{(\nu')^2/2}\, \mathrm{e}^{\mathrm{i}\pi\nu}\, \frac{\theta_2(\nu + \tau\nu'|\tau) + \mathrm{i}\theta_1(\nu + \tau\nu'|\tau)}{2\eta(\tau)} . \end{aligned} \tag{B.11}$$

[20] Remember that the values ν' and ν were defined by eqs. (4.7)–(4.8) only up to integer shifts.

Whenever zero-modes are present, one has to be careful. In those cases, the validity of the expressions in eq. (B.11) cannot be simply assumed. Fermion zero-modes are present in the untwisted R and $\mathrm{R}(-)^F$ sectors, and in the NS and $\mathrm{NS}(-)^F$ sectors twisted by an element of order two (i.e., such that $\nu' = 1/2$). The zero-modes $e^+ = \psi_0^i/\sqrt{2}$ and $e^- = \psi_0^{\bar{i}}/\sqrt{2}$, satisfy a Clifford algebra: $\{e^+ e^-\} = 1$. This gives a two-state spectrum containing $|\downarrow\rangle$, such that $e^-|\downarrow\rangle = 0$, and $|\uparrow\rangle = e^+|\downarrow\rangle$. In this basis, the rotation generator for the fermions, restricted to the zero modes and to rotations in the 1-2 plane (having set $\psi^i = \psi^1 + \mathrm{i}\psi^2$) reads simply $J = J_{12} = \frac{1}{2}[e^+, e^-] = \frac{1}{2}\sigma^3$. The action of g, namely a rotation by an angle $2\pi\nu$, is effected on the fermion zero-modes by $\exp(2\pi \mathrm{i}\nu J)$, that is by $\cos\pi\nu\mathbf{1} + \mathrm{i}\sin\pi\nu\,\sigma^3$. Since the fermion number operator $(-)^F$ is just $-\sigma^3$, the trace of g over the zero-mode sector with or without the insertion of the fermion number operator is given respectively by $-2\mathrm{i}\sin\pi\nu$ or by $2\cos\pi\nu$.

Thus, for instance, in the $\mathrm{R}(-)^F$ sector we get,

$$q^{1/12}\,(-2\mathrm{i}\sin\pi\nu)\prod_{n=1}^{\infty}(1-\mathrm{e}^{2\pi\mathrm{i}\nu}q^n)(1-\mathrm{e}^{-2\pi\mathrm{i}\nu}q^n) = -\mathrm{i}\,\frac{\theta_1(\nu|\tau)}{\eta(\tau)}\,, \tag{B.12}$$

In terms of the SO(2) (generalised) spinorial characters $\chi_{s,c}$, this translates into the expressions given in eq. (4.10) in the text.

Modular transformations We outline now the modular properties of the bosonic and fermionic characters computed above.

The modular properties of chiral blocks of orbifold theories were described, in the context of rational conformal field theories, in [53]; in these cases, under the action of the generators $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$, the chiral blocks χ_h^g transform as follows:

$$\chi_h^g \xrightarrow{S} \sigma(h|g)\,\chi_g^{h^{-1}} \tag{B.13}$$

$$\chi_h^g \xrightarrow{T} \mathrm{e}^{-\pi\mathrm{i}c/12}\,\tau_h\,\chi_h^{hg}\,. \tag{B.14}$$

The quantities $\sigma(h|g), \tau_h$ are related and encode the conformal dimension Δ_h of the twist field creating the vacuum of the h-twisted sector:

$$\mathrm{e}^{\pi\mathrm{i}\frac{c}{4}}\,\sigma(h|h)^{-1} = \tau_e\,(\tau_h)^2 = \mathrm{e}^{4\pi\mathrm{i}\Delta_h}\,. \tag{B.15}$$

Since free bosons do not generically build a rational CFT, it is not guaranteed a priori that the modular properties described above carry over unchanged. However, by explicitly carrying out the modular transformations relying on the properties of theta functions, one finds that they do.

The set of bosonic characters given in eq. (B.4) behaves under S as

$$\chi_h^{(X)\,g}(q') = \mathrm{i}\mathrm{e}^{2\pi\mathrm{i}\left(\nu\nu' - \frac{\nu+\nu'}{2}\right)}\,\chi_g^{(X)\,h^{-1}}(q)\,, \tag{B.16}$$

where $\tau' = -1/\tau$ and $q' = \exp(2\pi \mathrm{i}\tau')$, and we used eq. (C.6) or eq. (C.18). This corresponds to

$$\sigma(h|g) = \mathrm{e}^{2\pi\mathrm{i}(\nu'-1/2)(\nu-1/2)}\,. \tag{B.17}$$

Under the T generator, using theta function properties eq. (C.7) and setting $\tau'' = \tau + 1$, one has that

$$\chi_h^{(X)\,g}(q'') = \mathrm{e}^{-\mathrm{i}\pi/6}\mathrm{e}^{\pi\mathrm{i}\nu'(1-\nu')}\chi_h^{(X)\,hg}(q)\,. \tag{B.18}$$

Thus, by comparison with eq. (B.18) we get, since $c = 2$ for a complex boson,

$$\tau_h \equiv \mathrm{e}^{2\pi\mathrm{i}\Delta_h} = \mathrm{e}^{\pi\mathrm{i}\nu'(1-\nu')}\,. \tag{B.19}$$

The quantities $\sigma(h|g)$ and τ_h of eqs. (B.17), (B.18) correspond to the dimension of the twist field that creates the vacuum of the $\mathcal{H}_h$ twisted sector being

$$\Delta_h = \frac{1}{2}\nu'(1-\nu)\,, \tag{B.20}$$

in agreement with the expression of the zero-point energy of a complex boson in the h-twisted sector, eq. (B.5), as $a^{(X)} = c/24 - \Delta_h$.

We have to consider separately the characters in the untwisted sectors. In the absence of continuous momentum the untwisted characters are the $\hat{\chi}_e^g(q)$ of eq. (B.6), and the action of S, eq. (4.12) in the text, corresponds to

$$\widehat{\sigma}(e|g) = 2\sin\pi\nu\,. \tag{B.21}$$

Let us next consider the fermionic characters eq. (B.9). In the twisted sectors ($h \neq e$), and for h not of order two, under the S modular transformation we find, using eq. (C.5),

$$\chi_{\zeta',h}^{\zeta,g}(q') = \mathrm{e}^{-2\pi\mathrm{i}(\tilde{\nu}'+\zeta'-1/2)(\tilde{\nu}+\zeta-1/2)}\,\chi_{\zeta,g}^{\zeta',h^{-1}}(q)\,, \tag{B.22}$$

with $\tilde{\nu}$ being an integer shift of ν such that $\tilde{\nu} + 1/2 < 1$, in analogy with $\tilde{\nu}'$. This transformations exhibits phases

$$\sigma(\zeta',h|\zeta,g) = \mathrm{e}^{-2\pi\mathrm{i}(\tilde{\nu}'+\zeta'-1/2)(\tilde{\nu}+\zeta-1/2)}\,. \tag{B.23}$$

Under the T-modular transformation one finds

$$\chi_{\zeta',h}^{\zeta,g}(q'') = \mathrm{e}^{-\pi\mathrm{i}/12}\mathrm{e}^{\pi\mathrm{i}(\tilde{\nu}'+\zeta'-1/2)^2}\,\chi_{\zeta',h}^{\zeta+\zeta'\bmod 1,hg}(q)\,. \tag{B.24}$$

These expressions yield conformal weights of the spin-twist fields that agree with the 0-point energies of eq. (B.8):

$$\tau_{\zeta',h} \equiv \mathrm{e}^{2\pi\mathrm{i}\Delta_{\zeta',h}} = \mathrm{e}^{\pi\mathrm{i}(\tilde{\nu}'+\zeta'-1/2)^2}\,. \tag{B.25}$$

The quantities in eq. (B.23) and eq. (B.25) correctly satisfy the relation eq. (B.15), taking into account the conformal weight 1/8 of the SO(2) spin fields that map the NS vacuum to the R vacuum: $\tau_{0,e} = \exp(\pi \mathrm{i}/8)$.

In the untwisted case, as well as in the in the case in which h is of order two, the expressions are simpler than those in the generic cases. They turn out to be

$$\sigma(\zeta', e|\zeta, g) = \mathrm{e}^{-\pi \mathrm{i}(\zeta'-1/2)(\zeta-1/2)}\,, \tag{B.26}$$

i.e., only the $\mathrm{R}(-)^F$ sector acquires a $-\mathrm{i}$ factor. In the NS sectors twisted by $h \neq e$ with $h^2 = e$ we have the same behaviour, with

$$\sigma(\zeta', h|\zeta, g) = \sigma\left(\zeta' + \frac{1}{2} \bmod 1, e|\zeta, g\right) \qquad (h^2 = e)\,. \tag{B.27}$$

For various purposes, it may be more useful to express the action of the S-modular transformation in terms of the o, v, s, c characters. As such, the phases are collected in 4×4 matrices $S(h|g)$ acting on the vector $(\chi_a)^g_h$, with $a = v, o, s, c$:

$$(\chi_a)^g_h \xrightarrow{S} [S(h|g)]_a{}^b (\chi_b)^{h^{-1}}_g\,. \tag{B.28}$$

In terms of the phases of eq. (B.23), we have, for h generic,

$$S(h|g) = \frac{1}{2}\begin{pmatrix} \sigma(\frac{1}{2}, h|\frac{1}{2}, g) & \sigma(\frac{1}{2}, h|\frac{1}{2}, g) & \sigma(\frac{1}{2}, h|0, g) & \sigma(\frac{1}{2}, h|0, g) \\ \sigma(\frac{1}{2}, h|\frac{1}{2}, g) & \sigma(\frac{1}{2}, h|\frac{1}{2}, g) & -\sigma(\frac{1}{2}, h|0, g) & -\sigma(\frac{1}{2}, h|0, g) \\ -\sigma(0, h|\frac{1}{2}, g) & \sigma(0, h|\frac{1}{2}, g) & \sigma(0, h|0, g) & -\sigma(0, h|0, g) \\ -\sigma(0, h|\frac{1}{2}, g) & \sigma(0, h|\frac{1}{2}, g) & -\sigma(0, h|0, g) & \sigma(0, h|0, g) \end{pmatrix}. \tag{B.29}$$

In the untwisted case, the matrix takes a much simpler form. It coincides simply with the S matrix for the usual SO(2) chiral blocks, given in eq. (3.7) in the text (for $n = 1$): $S(e|g) = S_{(2)}$. An analogous expression is obtained when the twist h is of order 2, with $(\chi_o)^g_h \leftrightarrow (\chi_s)^g_e$ and $(\chi_v)^g_h \leftrightarrow (\chi_c)^g_e$.

C. Useful formulae

C.1 Theta functions

In terms of the quantities

$$\begin{aligned} q &= \exp(2\pi \mathrm{i}\tau) \\ z &= \exp(2\pi \mathrm{i}\nu)\,, \end{aligned} \tag{C.1}$$

we define the theta functions with characteristic $\left[{a \atop b}\right]$

$$\theta\left[{a \atop b}\right](v|\tau) = \sum_{n\in\mathbb{Z}} \exp\left\{\mathrm{i}\left(n - \frac{a}{2}\right)^2 \pi\tau + 2\pi \mathrm{i}\left(v - \frac{b}{2}\right)\left(n - \frac{a}{2}\right)\right\}. \tag{C.2}$$

These functions have several periodicity properties rather obvious from their definition eq. (C.2); in particular,

$$\theta[{}^{\ \ a}_{b-2m}](z|\tau) = \mathrm{e}^{-\pi \mathrm{i} a m}\theta[{}^a_b](z|\tau)\,, \qquad m \in \mathbb{Z}\,. \tag{C.3}$$

If the argument z of the theta-function is of the form $z = \nu + \tau\nu'$ it can be absorbed in the characteristic vector:

$$\theta[{}^a_b](\nu + \tau\nu'|\tau) = q^{-\nu'^2/2}\mathrm{e}^{\mathrm{i}\pi\nu'(b-2\nu)}\theta[{}^{a-2\nu'}_{b-2\nu}](0|\tau)\,. \tag{C.4}$$

Under the modular transformation $S : \tau \to \tau' = -1/\tau$ we have

$$\theta[{}^a_b](0|\tau) = (-\mathrm{i}\tau')^{1/2}\mathrm{e}^{\mathrm{i}\pi ab/2}\,\theta[{}^{\ b}_{-a}](0|\tau')\,. \tag{C.5}$$

Using eq. (C.4) we may also write

$$\theta[{}^a_b](\nu + \nu'\tau|\tau) = (-\mathrm{i}\tau')^{1/2}\mathrm{e}^{\mathrm{i}\pi ab/2}\,\mathrm{e}^{\mathrm{i}\pi((\nu')^2/\tau' + \nu^2\tau' - 2\nu'\nu)}\theta[{}^{\ b}_{-a}](-\nu' + \nu\tau'|\tau')\,. \tag{C.6}$$

Under the modular transformation $T : \tau \to \tau''$ we have

$$\theta[{}^a_b](\nu + \nu'\tau|\tau) = \mathrm{e}^{\mathrm{i}\pi\frac{a(a-2)}{4}}\,\theta[{}^{\ \ \ a}_{b-a+1}](\nu - \nu' + \nu'\tau''|\tau'')\,. \tag{C.7}$$

The theta-functions can be expressed as infinite products by making use of the "Jacobi triple product identity"

$$\prod_{n=0}^{\infty}(1 - q^{n+1})(1 - wq^{n+\frac{1}{2}})(1 - w^{-1}q^{n+\frac{1}{2}}) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}w^n\,. \tag{C.8}$$

Introducing the usual Jacobi notation

$$\begin{aligned}
\theta_1(v|\tau) &= \theta[{}^1_1](v|\tau)\,, \qquad & \theta_2(v|\tau) &= \theta[{}^1_0](v|\tau)\,,\\
\theta_3(v|\tau) &= \theta[{}^0_0](v|\tau)\,, \qquad & \theta_4(v|\tau) &= \theta[{}^0_1](v|\tau)\,,
\end{aligned} \tag{C.9}$$

one finds

$$\theta_1(v|\tau) = 2\exp\left(\frac{\pi\mathrm{i}\tau}{4}\right)\sin\pi v\prod_{m=1}^{\infty}(1 - q^m)(1 - zq^m)(1 - z^{-1}q^m) \tag{C.10}$$

$$\theta_2(v|\tau) = 2\exp\left(\frac{\pi\mathrm{i}\tau}{4}\right)\cos\pi v\prod_{m=1}^{\infty}(1 - q^m)(1 + zq^m)(1 + z^{-1}q^m) \tag{C.11}$$

$$\theta_3(v|\tau) = \prod_{m=1}^{\infty}(1 - q^m)(1 + zq^{m-1/2})(1 + z^{-1}q^{m-1/2}) \tag{C.12}$$

$$\theta_4(v|\tau) = \prod_{m=1}^{\infty}(1 - q^m)(1 - zq^{m-1/2})(1 - z^{-1}q^{m-1/2})\,. \tag{C.13}$$

The theta functions satisfy Jacobi's "abstruse identity"

$$\theta_3(0|\tau)^4 - \theta_2(0|\tau)^4 - \theta_4(0|\tau)^4 = 0\,. \tag{C.14}$$

Further,

$$\theta_1(0|\tau) = 0\,. \tag{C.15}$$

The Dedekind eta function is defined by

$$\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1-q^m)\,. \tag{C.16}$$

It has the modular transformation properties

$$\eta(\tau+1) = \exp(\mathrm{i}\pi/12)\eta(\tau) \tag{C.17}$$

and

$$\eta(-1/\tau) = (-\mathrm{i}\tau)^{1/2}\eta(\tau)\,. \tag{C.18}$$

C.2 Discrete groups

In the text we make repeatedly use of the orthogonality properties of the character table

$$\rho_I{}^\alpha \equiv \mathrm{tr}_{\mathcal{D}^I}\, g^{(\alpha)} \tag{C.19}$$

of a discrete group Γ. Above, we have denoted by $\mathcal{D}^I$ the irreducible representations of Γ; moreover, we denote by $\mathcal{C}^\alpha$ the conjugacy class of the element $g^{(\alpha)}$ (the traces are invariant under conjugation). The set of conjugacy classes is in one-to-one correspondence with the irreducible representations, so that the character table $\rho_I{}^\alpha$ is a square matrix. It enjoys the following orthogonality properties:

$$\begin{aligned} \frac{1}{|\Gamma|}\sum_\alpha n_\alpha\, (\rho_I{}^\alpha)^*\, \rho_J{}^\alpha &= \delta_{IJ} \\ \frac{1}{|\Gamma|}\sum_I (\rho_I{}^\alpha)^*\, \rho_I{}^\beta &= \frac{\delta^{\alpha\beta}}{n_\alpha}\,, \end{aligned} \tag{C.20}$$

where $|\Gamma|$ is the order of Γ and n_α is the number of elements of the conjugacy class $\mathcal{C}^\alpha$.

References

[1] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, *Nucl. Phys.* **B 438** (1995) 109 [hep-th/9410167].

[2] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124].

[3] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds*, *Nucl. Phys.* **B 261** (1985) 678.

[4] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds, 2*, *Nucl. Phys.* **B 274** (1986) 285.

[5] P.S. Aspinwall, *Enhanced gauge symmetries and K3 surfaces*, *Phys. Lett.* **B 357** (1995) 329 [`hep-th/9507012`].

[6] W. Nahm and K. Wendland, *A hiker's guide to K3: aspects of $n = (4,4)$ superconformal field theory with central charge $c = 6$*, `hep-th/9912067`.

[7] M.R. Douglas, *Enhanced gauge symmetry in M(atrix) theory*, *J. High Energy Phys.* **07** (1997) 004 [`hep-th/9612126`].

[8] D.-E. Diaconescu, M.R. Douglas and J. Gomis, *Fractional branes and wrapped branes*, *J. High Energy Phys.* **02** (1998) 013 [`hep-th/9712230`].

[9] J. Polchinski and Y. Cai, *Consistency of open superstring theories*, *Nucl. Phys.* **B 296** (1988) 91.

[10] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Adding holes and crosscaps to the superstring*, *Nucl. Phys.* **B 293** (1987) 83.

[11] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, *Loop corrections to superstring equations of motion*, *Nucl. Phys.* **B 308** (1988) 221.

[12] N. Ishibashi and T. Onogi, *Open string model building*, *Nucl. Phys.* **B 318** (1989) 239.

[13] P.D. Vecchia and A. Liccardo, *D-branes in string theory, 1*, `hep-th/9912161`.

[14] P.D. Vecchia and A. Liccardo, *D-branes in string theory, 2*, `hep-th/9912275`.

[15] B. Craps, *D-branes and boundary states in closed string theories*, `hep-th/0004198`.

[16] M.R. Gaberdiel, *Lectures on non-BPS Dirichlet branes*, *Class. and Quant. Grav.* **17** (2000) 3483 [`hep-th/0005029`].

[17] J. Dai, R.G. Leigh and J. Polchinski, *New connections between string theories*, *Mod. Phys. Lett.* **A 4** (1989) 2073.

[18] P. Hořava, *Strings on world sheet orbifolds*, *Nucl. Phys.* **B 327** (1989) 461.

[19] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [`hep-th/9510017`].

[20] G. Pradisi and A. Sagnotti, *Open string orbifolds*, *Phys. Lett.* **B 216** (1989) 59.

[21] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[22] D.-E. Diaconescu and J. Gomis, *Fractional branes and boundary states in orbifold theories*, *J. High Energy Phys.* **10** (2000) 001 [`hep-th/9906242`].

[23] M.R. Gaberdiel and J. Bogdan Stefanski, *Dirichlet branes on orbifolds*, *Nucl. Phys.* **B 578** (2000) 58 [`hep-th/9910109`].

[24] T. Takayanagi, *String creation and monodromy from fractional D-branes on ALE spaces*, *J. High Energy Phys.* **02** (2000) 040 [`hep-th/9912157`].

[25] F. Roose, *Boundary states and non-abelian orbifolds*, `hep-th/0002129`.

[26] M. Billó, B. Craps and F. Roose, *On D-branes in type-0 string theory*, *Phys. Lett.* **B 457** (1999) 61 [`hep-th/9902196`].

[27] W. Lerche, D. Lust and A.N. Schellekens, *Chiral four-dimensional heterotic strings from selfdual lattices*, *Nucl. Phys.* **B 287** (1987) 477.

[28] H. Kawai, D.C. Lewellen and S.H. H. Tye, *Construction of fermionic string models in four-dimensions*, *Nucl. Phys.* **B 288** (1987) 1.

[29] M. Bianchi and A. Sagnotti, *On the systematics of open string theories*, *Phys. Lett.* **B 247** (1990) 517.

[30] O. Bergman and M.R. Gaberdiel, *A non-supersymmetric open-string theory and S-duality*, *Nucl. Phys.* **B 499** (1997) 183 [`hep-th/9701137`].

[31] I.R. Klebanov and A.A. Tseytlin, *D-branes and dual gauge theories in type-0 strings*, *Nucl. Phys.* **B 546** (1999) 155 [`hep-th/9811035`].

[32] E.R. Sharpe, *Discrete torsion*, `hep-th/0008154`.

[33] M.R. Gaberdiel, *Discrete torsion orbifolds and D-branes*, *J. High Energy Phys.* **11** (2000) 026 [`hep-th/0008230`].

[34] P.S. Aspinwall and M.R. Plesser, *D-branes, discrete torsion and the mckay correspondence*, `hep-th/0009042`.

[35] P.S. Aspinwall, *A note on the equivalence of Vafa's and Douglas's picture of discrete torsion*, *J. High Energy Phys.* **12** (2000) 029 [`hep-th/0009045`].

[36] B. Feng, A. Hanany, Y.-H. He and N. Prezas, *Discrete torsion, non-abelian orbifolds and the schur multiplier*, *J. High Energy Phys.* **01** (2001) 033 [`hep-th/0010023`].

[37] C. Vafa, *Modular invariance and discrete torsion on orbifolds*, *Nucl. Phys.* **B 273** (1996) 592.

[38] M.R. Douglas, *D-branes and discrete torsion*, `hep-th/9807235`.

[39] J. McKay, *Graphs, singularities and finite groups* in *Proc. Symp. Pure Path*, A.M.S. **37** (1980) 183.

[40] E. Verlinde, *Fusion rules and modular transformations in 2-d conformal field theory*, *Nucl. Phys.* **B 300** (1988) 360.

[41] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186].

[42] J. Fuchs and C. Schweigert, *Branes: from free fields to general backgrounds*, *Nucl. Phys.* **B 530** (1998) 99 [hep-th/9712257].

[43] M. Billó, P.D. Vecchia and D. Cangemi, *Boundary states for moving D-branes*, *Phys. Lett.* **B 400** (1997) 63 [hep-th/9701190].

[44] P.D. Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, *Classical p-branes from boundary state*, *Nucl. Phys.* **B 507** (1997) 259 [hep-th/9707068].

[45] J. Polchinski, *String theory*, 2 volumes, Cambridge University Press, 1998.

[46] M. Billó, P. Di Vecchia, M. Frau, A. Lerda, I. Pesando, R. Russo and S. Sciuto, *Microscopic string analysis of the D0-D8 brane system and dual RR states*, *Nucl. Phys.* **B 526** (1998) 199 [hep-th/9802088].

[47] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, *Nucl. Phys.* **B 570** (2000) 525 [hep-th/9908036].

[48] J.A. Harvey, S. Kachru, G. Moore and E. Silverstein, *Tension is dimension*, *J. High Energy Phys.* **03** (2000) 001 [hep-th/9909072].

[49] J. Polchinski, *Tensors from K3 orientifolds*, *Phys. Rev.* **D 55** (1997) 6423 [hep-th/9606165].

[50] M.R. Douglas, B.R. Greene and D.R. Morrison, *Orbifold resolution by D-branes*, *Nucl. Phys.* **B 506** (1997) 84 [hep-th/9704151].

[51] W. Lerche, *On a boundary CFT description of nonperturbative $N = 2$ Yang-Mills theory*, hep-th/0006100.

[52] W. Lerche, C.A. Lutken and C. Schweigert, *D-branes on ALE spaces and the ADE classification of conformal field theories*, hep-th/0006247.

[53] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, *The operator algebra of orbifold models*, *Comm. Math. Phys.* **123** (1989) 485.

[54] M.R. Douglas and G. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[55] C.V. Johnson and R.C. Myers, *Aspects of type-IIB theory on ALE spaces*, *Phys. Rev.* **D 55** (1997) 6382 [hep-th/9610140].

[56] P.B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, *J. Diff. Geom.* **29** (1989) 665; *A Torelli type theorem for gravitational instantons*, *J. Diff. Geom.* **29** (1989) 685.

[57] C.V. Johnson, *D-brane primer*, hep-th/0007170.

[58] H. Garcia-Compean, *D-branes in orbifold singularities and equivariant K-theory*, *Nucl. Phys.* **B 557** (1999) 480 [hep-th/9812226].

[59] M.R. Douglas and B. Fiol, *D-branes and discrete torsion, 2*, hep-th/9903031.

[60] M. Berkooz, M.R. Douglas and R.G. Leigh, *Branes intersecting at angles*, *Nucl. Phys.* **B 480** (1996) 265 [hep-th/9606139].

[61] M. Berkooz and M.R. Douglas, *Five-branes in M(atrix) theory*, *Phys. Lett.* **B 395** (1997) 196 [hep-th/9610236].

[62] D. Anselmi, M. Billó, P. Fre, L. Girardello and A. Zaffaroni, *ALE manifolds and conformal field theories*, *Int. J. Mod. Phys.* **A 9** (1994) 3007 [hep-th/9304135].

[63] M. Reid, *La correspondance de McKay*, Séminaire Bourbaki, 52éme année, n. 867 [math.AG/9911165].

[64] M. Bertolini, P. D. Vecchia, M. Frau, A. Lerda, R. Marotta and I. Pesando, *Fractional D-branes and their gauge duals*, hep-th/0011077.

[65] K. Wendland, *Consistency of orbifold conformal field theories on K3*, hep-th/0010281.

[66] O. Bergman and M.R. Gaberdiel, *Non-BPS states in heterotic-type-IIA duality*, *J. High Energy Phys.* **03** (1999) 013 [hep-th/9901014].

[67] K. Dasgupta and S. Mukhi, *Brane constructions, fractional branes and anti-de Sitter domain walls*, *J. High Energy Phys.* **07** (1999) 008 [hep-th/9904131].